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CONTENTS

Vol. 48, No. 1 JANUARY, 1942 Whole No. 508

Page

Letter from Marston Morse 1

The October Meeting in New York. By T. R. HOLLCROFT 3

The November Meeting in Manhattan. By W. L. AYRES 6

The November Meeting in Pasadena. By T. M. PUTNAM 8

Book Reviews 10

Notes 22

Abstracts of Papers

Algebra and Theory of Numbers 29

Analysis 35

Applied Mathematics 44

Geometry 48

Logic and Foundations 49

Statistics and Probability 52

Topology 52

New Publications 56

CONTENTS

Vol. 48, No. 2 FEBRUARY, 1942 Whole No. 509

Page

The problem of Bolza in the calculus of variations. By M. R. HESTENES 57

The analysis of linear transformations. By F. J. MURRAY 76

Conformal geometry of one-parameter families of curves. By ROBERT COLEMAN, JR. 94

Some theorems on subseries. By J. D. HILL 103

A formula for the direct product of crossed product algebras. By SAUNDERS MACLANE and O. F. G. SCHILLING 108

Connected and disconnected plane sets and the functional equation $f(x)+f(y)=f(x+y)$. By F. B. JONES 115

A remark on the cardinal of limit spaces. By W. S. SNYDER ... 121

A note on the representation by polygonal numbers. By L. W. GRIFFITHS 122

Hausdorff methods of summation which include all of the Cesàro methods. By H. L. GARABEDIAN 124

A characterization of the radical of an algebra. By SAM PERLIS. 128

An extension of a theorem of Witt. By B. W. JONES 133

Expansions in series of non-orthogonal functions. By R. V. CHURCHILL 143

On convex sets in linear normed spaces. By TRUMAN BOTTS .. 150

Forced oscillations of continuous dynamical systems. By W. H. INGRAM 153

Note on the coefficients of overconvergent power series. By J. L. WALSH 163

Correction to "Totally geodesic Einstein spaces." By AARON FIALKOW 167

CONTENTS

Vol. 48, No. 3 MARCH, 1942 Whole No. 510

Page

Summer Meeting at Chicago. Conference Reports 169

The Annual Meeting of the Society. By T. R. HOLLCROFT.... 183

James Waterman Glover—In Memoriam. By PETER FIELD... 199

Book Reviews 201

Notes 206

Abstracts of Papers

Algebra and Theory of Numbers 210

Analysis 212

Applied Mathematics 216

Geometry 216

Logic and Foundations 219

Statistics and Probability 220

Topology 220

New Publications 222

CONTENTS

Vol. 48, No. 4 APRIL, 1942 Whole No. 511

Page

Statistical genetics and evolution. By SEWALL WRIGHT..... 223

Mathematical problems connected with the bending and buckling of elastic plates. By J. J. STOKER..... 247

Arithmetic of ordinals with applications to the theory of ordered Abelian groups. By P. W. CARRUTH..... 262

A characterization of absolute neighborhood retracts. By R. H. FOX..... 271

Resolution of boundary problems by the use of a generalized convolution. By R. C. F. BARTELS and R. V. CHURCHILL.... 276

Roots of certain classes of polynomials. By LOUIS WEISNER... 283

Generalized Laplace integrals. By R. P. BOAS, JR..... 286

Classes of maximum numbers associated with two symmetric equations. By H. A. SIMMONS..... 295

On a family of Fourier transforms. By AUREL WINTNER..... 304

Everywhere dense subgroups of Lie groups. By P. A. SMITH... 309

Vector spaces over rings. By C. J. EVERETT..... 312

Pseudo-conformal geometry: functions of two complex variables. By EDWARD KASNER and JOHN DECICCO..... 317

On integral functions of integral or zero order. By S. M. SHAH. 329

CONTENTS

Vol. 48, No. 5 MAY, 1942 Whole No. 512

Page

The February Meeting in New York. By T. R. HOLLCROFT.... 335

Book Reviews..... 338

Notes..... 346

Abstracts of Papers

Algebra and Theory of Numbers..... 355

Analysis..... 361

Applied Mathematics..... 369

Geometry..... 371

Logic and Foundations..... 373

Statistics and Probability..... 374

Topology..... 374

New Publications..... 378

CONTENTS

Vol. 48, No. 6 JUNE, 1942 Whole No. 513

Page

Trends in research: The analytic number theory. By HANS RADEMACHER 379

Normal varieties and birational correspondences. By OSCAR ZARISKI 402

On differentiation of integrals and approximate continuity. By ARTHUR ROSENTHAL 414

A note on Hilbert's operator. By H. KOBER 421

The behavior of certain Stieltjes continued fractions near the singular line. By H. S. WALL 427

On local convexity in Hilbert space. By I. J. SCHOENBERG 432

On 3-dimensional manifolds. By C. E. CLARK 437

A linear transformation whose variables and coefficients are sets of points. By S. T. SANDERS, JR. 440

A theorem on Lie groups. By DEANE MONTGOMERY and LEO ZIPPIN 448

A note on complementation in lattices. By L. R. WILCOX 453

Sufficient conditions that polynomials in several variables be positive. By I. E. PERLIN 458

On the mapping of n quadratic forms. By L. L. DINES 467

Measure and other properties of a Hamel basis. By F. B. JONES 472

On the composition of fields. By CLAUDE CHEVALLEY 482

Note on the closure of orthogonal polynomials. By J. SHOHAT .. 488

CONTENTS

Vol. 48, No. 7 JULY, 1942 Whole No. 514

Page

The April Meeting in New York. By T. R. HOLLCROFT 491

The April Meeting in Berkeley. By A. D. MICHAL 495

The April Meeting in Chicago. By W. L. AYRES 496

Robert Henderson—In Memoriam. By R. D. MURPHY and R. G. D. RICHARDSON 504

Book Reviews 506

Notes 512

Abstracts of papers

Algebra and Theory of Numbers 518

Analysis 519

Applied Mathematics 522

Geometry 522

Statistics and Probability 523

Topology 523

New Publications 525

CONTENTS

Vol. 48, No. 8 AUGUST, 1942 Whole No. 515

Page

Traveling waves on electric power systems. By L. V. BEWLEY.. 527

Some new methods of solution of two-dimensional problems in elasticity. By I. S. SOKOLNIKOFF..... 539

On equivalence of certain types of series of orthonormal functions. By G. H. PEEBLES..... 556

A mapping characterization of Peano spaces. By O. G. HARROLD, JR..... 561

Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas. By TOMLINSON FORT... 567

Factorization of differential ideals. By H. J. RIBLET..... 575

On the iteration of linear homogeneous transformations. By ARNOLD DRESDEN..... 577

A sufficient condition for Cesàro summability. By G. D. NICHOLS..... 580

On the theory of the tetrahedron. By N. A. COURT..... 583

Euclidean concomitants of the ternary cubic. By T. L. WADE.. 589

Generalization of a theorem of Korous on the bounds of orthonormal polynomials. By DUNHAM JACKSON..... 602

Approximation of continuous functions by means of lacunary polynomials. By BERNARD DIMSDALE..... 608

Generalized Fischer groups and algebras. By R. H. BRUCK.... 618

A new proof of the cyclic connectivity theorem. By W. L. AYRES 627

Inverses and zero-divisors. By REINHOLD BAER..... 630

CONTENTS

Vol. 48, No. 9 SEPTEMBER, 1942 Whole No. 516

PART 1

Page

Book Reviews 639

Notes 659

Abstracts of Papers

Algebra and Theory of Numbers 665

Analysis 666

Applied Mathematics 669

Geometry 670

Statistics and Probability 673

Topology 675

New Publications 678

CONTENTS

Vol. 48, No. 10 OCTOBER, 1942 Whole No. 517

Page

Some topics in the arithmetic of polynomials. By L. CARLITZ... 679

Differential equations with general boundary conditions. By W. M. WHYBURN..... 692

On the logarithmic means of rearranged partial sums of a Fourier series. By OTTO SZÁSZ..... 705

The basic analogue of Kummer's theorem. By J. A. DAUM... 711

Perron integrals. By R. L. JEFFERY..... 714

On Perron integration. By E. J. MCSHANE..... 718

On the least primitive root of a prime. By LOO-KENG HUA... 726

On the least solution of Pell's equation. By LOO-KENG HUA... 731

A note on the Diophantine problem of finding four biquadrates whose sum is a biquadrate. By J. O. PATTERSON..... 736

On the inversion of the q-series associated with Jacobian elliptic functions. By A. N. LOWAN, G. BLANCH and W. HORENSTEIN. 737

Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula. By A. N. LOWAN, NORMAN DAVIDS and ARTHUR LEVENSON..... 739

Some remarks on relatively complete fields. By IRVING KAPLANSKY and O. F. G. SCHILLING..... 744

A differential geometry problem using tensor analysis. By A. H. SPRAGUE..... 747

On the nilpotency of the radical of a ring. By RICHARD BRAUER. 752

On an elementary analogue of the Riemann-Mangoldt formula. By AUREL WINTNER..... 759

A note on the lower semi-continuity of double integrals in the parametric form. By WILLIAM SCOTT..... 763

A non-involutorial space transformation associated with a $Q_{1,n}$ congruence. By M. L. VEST..... 767

Hausdorff means included between $(C, 0)$ and $(C, 1)$. By H. J. GREENBERG and H. S. WALL..... 774

Whirl-similitudes, euclidean kinematics, and non-euclidean geometry. By J. M. FELD..... 783

CONTENTS

Vol. 48, No. 11 NOVEMBER, 1942 Whole No. 518

Page

A Proposed Journal of Applied Mathematics. By E. R. Hedrick 791

Occupational Classification of Mathematicians. By Marston Morse and J. R. Kline 792

Occupational Bulletin No. 23 795

Occupational Bulletin No. 10 797

The Summer Meeting in Poughkeepsie. By T. R. Hollcroft 800

Book Reviews 810

Notes 819

Abstracts of Papers

Algebra and Theory of Numbers 825

Analysis 827

Applied Mathematics 831

Geometry 833

Logic and Foundations 834

Statistics and Probability 834

Topology 835

New Publications 838

CONTENTS

Vol. 48, No. 12 DECEMBER, 1942 Whole No. 519

Page

Entire functions of exponential type. By R. P. BOAS, JR. 839
Some notes on an expansion theorem of Paley and Wiener. By R. J. DUFFIN and J. J. EACHUS 850
A generalization of the polar representation of nonsingular matrices. By JOHN WILLIAMSON 856
Remarks on regularity of methods of summation. By G. E. FORSYTHE and A. C. SCHAEFFER 863
Projections of the prime-power abelian group of order p^m and type $(m-1, 1)$. By R. A. BEAUMONT 866
Symmetric differential expressions. By H. J. RIBLET 871
Non-involutorial space transformations associated with a linear congruence. By M. L. VEST 874
The measure of the critical values of differentiable maps. By ARTHUR SARD 883
The radical of a non-associative algebra. By A. A. ALBERT 891
An arithmetical identity for the form $ab-c^2$. By W. H. GAGE 898
Dual geodesics on a surface. By C. E. SPRINGER 901
Generators of permutation groups simply isomorphic with $LF(2, p^n)$. By F. A. LEWIS 907
On an inequality of Seidel and Walsh. By L. H. LOOMIS 908
The Cauchy theorem for functions on closed sets. By P. T. MAKER 912
On value regions of continued fractions. By WALTER LEIGHTON and W. J. THRON 917
A table of coefficients for numerical differentiation. By A. N. LOWAN, H. E. SALZER and ABRAHAM HILLMAN 920
On majorants of subharmonic and analytic functions. By FRANTIŠEK WOLF 925
Homogeneous and nonhomogeneous Diophantine equations. By A. A. AUCOIN 933
Vector analogues of Morera's theorem. By E. F. BECKENBACH. 937
On the interiority of real functions. By G. T. WHYBURN 942
The modular space determined by a positive function. By R. W. BARNARD and H. H. GOLDSTINE 946
A correction to "On the iteration of linear homogeneous transformations." By ARNOLD DRESDEN 949
An addition to "Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas." By TOMLINSON FORT 949
Errata, Volume 48 950
Index of Volume 48 951

BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

LETTER FROM MARSTON MORSE

Institute for Advanced Study
Princeton, New Jersey
November 8, 1941.

The object of this letter is to give to interested mathematicians a report concerning developments in the problem of the proper use of mathematicians of the draft age.

At a July meeting of the Roster of Scientific and Specialized Personnel[1] in Washington these matters were discussed at length and a number of those present, including the writer, urged that the Roster is the body best equipped to take up the problem of the proper use of scientists. It should be understood at the beginning of this letter that the Roster has no power to defer scientists, but is concerned only with making recommendations as to whether or not a man may be regarded as "necessary" in the sense of the law.

In order to make proper recommendations it was necessary that the Roster obtain considerable additional information concerning men of draft age on its lists. The matter of procedure has now been systematized. Any man who is on the Roster and who finds himself likely to be called may write the Roster for a standard questionnaire. Among other things this questionnaire seeks to find out the man's present occupation and scientific status and the names of references. These references and the man's employer are then sent questionnaires calculated to assist in determining whether the man may be regarded as "necessary."

The Roster than calls in men in the various scientific fields to review the assembled data and advise the Roster on each individual case. The mathematicians were the first so treated, and Murnaghan, Hotelling and Morse recently served as consultants for the Roster. The Roster had assembled excellent data bearing on 300 cases, each of which was carefully reviewed by each of the three mathematicians. In general we tried to follow the resolutions concerning deferment

[1] The Roster of Scientific and Specialized Personnel, Washington, D. C., is a sufficient address.

which were voted at the recent Chicago meeting of the mathematicians and are published in the Bulletin.

The Roster makes its recommendations to the Selective Service Headquarters, which in turn *may* pass on the recommendations to the local boards. The local boards are free to follow the recommendations or not. We shall know more about the practice a little later.

In case a man is inducted, the Roster acts in an advisory capacity to the personnel department of the Adjutant General's office, offering information which will aid the personnel officers in properly using the scientists on the Roster.

In addition to making recommendations as to individual men, the Roster also seeks information concerning shortages in various scientific fields and passes on this information to the Selective Service Headquarters and the Labor Office. Our Committee on the Supply and Demand for Mathematicians will materially aid the Roster in this respect, and will help to clarify the situation.

In conclusion I wish to recommend that problems of deferment or use of scientists on the Roster be taken up with the Roster whenever they arise. With the advice of consultants in the various fields, the Roster is the group best organized and equipped to handle these questions.

Very truly yours,
MARSTON MORSE

THE OCTOBER MEETING IN NEW YORK

The three hundred eighty-first meeting of the American Mathematical Society was held at Columbia University on Saturday, October 25, 1941. The attendance included the following one hundred sixty-seven members of the Society:

A. A. Albert, G. K. Anderson, R. L. Anderson, Lawrence Annenberg, R. G. Archibald, L. A. Aroian, S. F. Barber, E. F. Beckenbach, M. F. Becker, Gertrude Blanch, H. F. Bohnenblust, S. G. Bourne, C. B. Boyer, A. T. Brauer, H. W. Brinkmann, H. C. Brodie, A. B. Brown, R. S. Burington, Jewell H. Bushey, S. S. Cairns, D. E. Christie, J. A. Clarkson, M. D. Clement, R. M. Cohn, T. F. Cope, Richard Courant, Ruth Crucet, J. H. Curtiss, M. D. Darkow, Norman Davids, D. R. Davis, Jesse Douglas, Arnold Dresden, Jacques Dutka, J. E. Eaton, M. L. Elbeback, Paul Erdös, F. H. Escott, J. M. Feld, Aaron Fialkow, W. B. Fite, Edward Fleisher, A. I. Forsythe, J. S. Frame, K. O. Friedrichs, H. P. Geiringer, B. P. Gill, Saul Gorn, L. M. Graves, M. C. Gray, C. C. Grove, E. J. Gumbel, Margaret Gurney, W. W. Gutzman, D. W. Hall, F. C. Hall, N. A. Hall, Hans-Karl Hammer, D. C. Harkin, M. H. Heins, Edward Helly, Olaf Helmer, E. H. C. Hildebrandt, L. S. Hill, Banesh Hoffmann, T. R. Hollcroft, M. W. Hopkins, Harold Hotelling, E. M. Hull, R. P. Isaacs, S. A. Joffe, R. A. Johnson, B. W. Jones, Shizuo Kakutani, G. K. Kalisch, Edward Kasner, L. S. Kennison, A. R. Kirby, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, Jack Laderman, M. E. Ladue, Solomon Lefschetz, B. A. Lengyel, Howard Levi, Marie Litzinger, E. R. Lorch, L. A. Lorch, Jack Lorell, A. N. Lowan, Eugene Lukacs, L. A. MacColl, C. C. MacDuffee, H. F. MacNeish, Dorothy Maharam, P. T. Maker, A. J. Maria, D. H. Maria, Walther Mayer, H. L. Mintzer, A. K. Mitchell, Don Mittleman, Morris Monsky, Marston Morse, F. J. Murray, C. A. Nelson, O. E. Neugebauer, J. C. Oxtoby, Everett Pitcher, E. L. Post, Walter Prenowitz, M. H. Protter, R. G. Putnam, H. A. Rademacher, C. J. Rees, M. S. Rees, R. W. Rempfer, H. J. Riblet, Moses Richardson, M. S. Robertson, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Arthur Sard, L. J. Savage, Rubin Schatten, Henry Scheffé, I. J. Schoenberg, L. W. Sheridan, Seymour Sherman, Max Shiffman, L. P. Siceloff, James Singer, L. L. Smail, D. M. Smiley, M. F. Smiley, P. A. Smith, Ernst Snapper, Virgil Snyder, Abraham Spitzbart, Wolfgang Sternberg, J. J. Stoker, W. C. Strodt, Alvin Sugar, M. M. Sullivan, Fred Supnick, J. D. Tamarkin, R. M. Thrall, W. R. Transue, W. J. Trjitzinsky, J. W. Tukey, J. L. Vanderslice, Oswald Veblen, G. C. Webber, Louis Weisner, G. W. Whaples, O. L. Wheeler, H. S. White, Hassler Whitney, John Williamson, Audrey Wishard, Jacob Wolfowitz, M. A. Woodbury, Leo Zippin, Antoni Zygmund.

The meeting opened in the morning with a general session for contributed papers at which Professor W. J. Trjitzinsky presided. President Marston Morse presided at the afternoon session which began with a short business meeting for consideration of the amendment of Section 9 of Article VII of the By-Laws proposed by the Council. The amendment was adopted. The section as amended is given below:

Section 9. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life

member shall have for life the status and privileges of an ordinary member, without further payment of dues. After October 25, 1941, no additional applications for life memberships will be accepted.

After the business meeting Professor I. J. Schoenberg gave an address entitled *Some analytical aspects of isometric imbedding.*

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 9 were read at the morning session; those numbered 10 to 27 were read by title. Dr. Bers was introduced by Dr. A. M. Gelbart.

1. Rubin Schatten: *On the direct product of Banach spaces.* (Abstract 47-11-476.)

2. Walter Prenowitz: *Descriptive geometries as multigroups.* (Abstract 47-11-481.)

3. E. F. Beckenbach: *On the analytic prolongation of a minimal surface.* (Abstract 47-9-418.)

4. S. S. Cairns: *The space of a variable geodesic complex on a sphere.* (Abstract 47-9-438.)

5. J. H. Curtiss: *On the distribution of the quotient of two chance variables.* (Abstract 47-9-434.)

6. E. J. Gumbel: *Simple tests for given statistical hypotheses.* (Abstract 47-11-485.)

7. Arthur Korn: *On the problem of pulsating spheres in a liquid.* (Abstract 47-11-478.)

8. Ernst Snapper: *Co-maximal linear sets and products of linear sets.* (Abstract 47-11-460.)

9. B. W. Jones: *The number of classes in related genera of quadratic forms.* (Abstract 48-1-13.)

10. Lipman Bers: *A convergence theorem for analytic functions of two variables.* (Abstract 47-11-467-*t*.)

11. D. G. Bourgin: *Linear topological spaces.* (Abstract 47-9-437-*t*.)

12. A. B. Brown: *Independent parameters for sets of functions.* (Abstract 47-11-468-*t*.)

13. S. S. Cairns: *Topological mapping of a Brouwer 4-manifold on an analytic Riemannian 4-manifold.* (Abstract 47-9-439-*t*.)

14. A. D. Campbell: *On the application of an algebra of sets to group theory.* (Abstract 47-11-455-*t*.)

15. George Comenetz: *Isotropic curves on a surface.* (Abstract 47-11-479-*t*.)

16. J. J. DeCicco: *Bi-isothermal systems of curves.* (Abstract 47-11-480-*t*.)

17. Howard Levi: *A characterization of polynomial rings by means of order relations.* (Abstract 47-11-458-*t*.)

18. A. N. Lowan, Norman Davids and Arthur Levinson: *Tables for Gauss' mechanical quadrature formula.* (Abstract 47-9-413-*t*.)

19. A. N. Lowan and William Horenstein: *On the function* $H(m, a, x) = \exp(-ix)F(m+1-ia, 2m+2, 2ix)$. (Abstract 47-11-470-*t*.)

20. R. E. O'Connor: *A theorem of Diophantine approximation and an application to the values of a linear form.* (Abstract 47-11-462-*t*.)

21. R. E. O'Connor: *Representation of integers by power-products of two integers.* (Abstract 47-11-463-*t*.)

22. R. E. O'Connor: *Representation of integers by power-products of two real numbers.* (Abstract 47-11-464-*t*.)

23. Harry Pollard: *Legendre polynomials and the Hausdorff moment problem.* (Abstract 47-9-400-*t*.)

24. Maxwell Reade and E. F. Beckenbach: *Square averages and a class of harmonic polynomials.* (Abstract 47-11-474-*t*.)

25. J. F. Ritt: *Complete difference ideals.* (Abstract 47-11-475-*t*.)

26. A. R. Schweitzer: *Concerning general abstract relational spaces.* III. (Abstract 47-9-430-*t*.)

27. H. S. Wall: *The behavior of certain Stieltjes continued fractions near the singular line.* (Abstract 47-11-477-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN MANHATTAN

The three hundred eighty-second meeting of the American Mathematical Society was held at Kansas State College, Manhattan, Kansas, on Friday and Saturday, November 21–22, 1941. The attendance was about eighty including the following forty-seven members of the Society:

A. A. Albert, M. N. Arnoldy, W. L. Ayres, R. W. Babcock, Wealthy Babcock, F. L. Black, L. M. Blumenthal, W. C. Brenke, C. C. Camp, R. D. Daugherty, H. P. Doole, Paul Eberhart, W. E. Ferguson, Cornelius Gouwens, O. H. Hamilton, J. J. L. Hinrichsen, D. L. Holl, W. C. Janes, H. E. Jordan, D. E. Kibbey, E. H. Larguier, P. E. Lewis, A. T. Lonseth, J. V. McKelvey, Anna Marm, U. G. Mitchell, T. A. Mossman, G. C. Munro, E. N. Oberg, R. S. Pate, T. A. Pierce, P. S. Pretz, G. B. Price, P. G. Robinson, D. H. Rock, L. L. Runge, G. E. Schweigert, C. E. Sealander, M. E. Shanks, D. T. Sigley, E. R. Smith, G. W. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, L. E. Ward, J. W. T. Youngs.

The meeting opened Friday afternoon in Mathematics Hall with a session for short papers, Professor W. T. Stratton presiding. Following this, Professor G. B. Price gave an address entitled *Linear extensions*. Professor L. M. Blumenthal presided at this lecture. On Saturday morning in Willard Hall with Professor C. C. Camp presiding, Professor A. A. Albert gave an address with the title *Some mathematical aspects of cryptography*. The meeting closed with a second session of contributed papers with Dean E. B. Stouffer presiding.

The dinner on Friday evening was held at the Country Club on a hill with a fine view of Manhattan and the surrounding country. The number attending the dinner was eighty-two. Professor W. T. Stratton presided at the dinner. After musical selections by Miss Hilda Grossman, President F. D. Farrell of Kansas State College welcomed the mathematicians and Associate Secretary W. L. Ayres responded for the Society. Professor U. G. Mitchell spoke on mathematics as an intellectual invention and urged the training of mathematicians for the social sciences. Dean R. W. Babcock talked on mathematical contemporaries urging that students be taught to look on mathematics as a living science and be told of living mathematicians.

Titles and cross references to the abstracts of the papers read at this meeting follow below. Papers 1 to 5 were read Friday afternoon, papers 6 to 10 on Saturday morning, papers 11 to 13, whose abstract numbers are followed by the letter *t*, were read by title. Mr. R. H. Tripp and Mr. A. E. Engelbrecht were introduced by Professor D. L. Holl.

1. D. L. Holl: *Consolidation of elastic earth layers.* (Abstract 48-1-68.)

2. A. A. Albert: *Quadratic forms permitting composition.* (Abstract 48-1-1.)

3. J. W. T. Youngs: *Concerning arc-curves and basic sets.* (Abstract 47-9-453.)

4. M. E. Shanks: *Irreducible transformations of the discontinuum into continua.* Preliminary report. (Abstract 48-1-97.)

5. J. J. L. Hinrichsen: *Bounds for the libration points in a restricted problem of n bodies.* (Abstract 48-1-67.)

6. L. M. Blumenthal: *Metric characterization of n-dimensional elliptic space* $\mathcal{E}_{nr}$. Preliminary report. (Abstract 48-1-75.)

7. R. S. Pate: *Functional homomorphisms.* I. Preliminary report. (Abstract 48-1-17.)

8. G. C. Munro: *Systems of linear differential equations with constant coefficients.* (Abstract 48-1-45.)

9. R. H. Tripp: *Bending of a thin plate having the form of a parallelogram.* (Abstract 48-1-73.)

10. A. E. Engelbrecht: *Circular plates with large deflections.* (Abstract 48-1-63.)

11. C. V. Robinson: *Spherical theorems of Helly type and congruence indices of spherical caps.* (Abstract 47-11-482-*t*.)

12. R. H. Bruck and T. L. Wade: *The number of independent components of the tensor* ${}_{[\alpha]}T_{i_1\cdots i_p}$. (Abstract 47-11-454-*t*.)

13. Isaac Opatowski: *On the theory of lethal irradiation of micro-örganisms.* II. (Abstract 48-1-69-*t*.)

W. L. AYRES,
Associate Secretary

THE NOVEMBER MEETING IN PASADENA

The three hundred eighty-third meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California, on November 22, 1941. The attendance was about seventy, including the following thirty-eight members of the Society:

O. W. Albert, L. D. Ames, R. S. Arthur, H. M. Bacon, Harry Bateman, Clifford Bell, J. L. Brenner, F. A. Butter, P. H. Daus, G. C. Evans, G. E. Forsythe, W. H. Glenn, H. J. Hamilton, Einar Hille, P. G. Hoel, C. G. Jaeger, Glenn James, D. H. Lehmer, R. J. Levit, S. H. Levy, G. F. McEwen, G. W. Mackey, A. D. Michal, W. T. Puckett, W. C. Roberts, G. E. F. Sherwood, D. V. Steed, J. D. Swift, P. M. Swingle, Gabor Szegö, A. E. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, Morgan Ward, W. M. Whyburn, E. R. Worthington, Max Zorn.

General sessions were held in the morning and in the afternoon, the sessions opening at 10 A.M. and 2 P.M., respectively. The morning session was presided over at different times by Professors T. Y. Thomas, Gabor Szegö, and Harry Bateman, and the afternoon session by Professor G. C. Evans.

By invitation of the Program Committee, Professor W. M. Whyburn of the University of California, Los Angeles, delivered an hour address on *Differential systems with general boundary conditions.*

Following the meeting Professor and Mrs. A. D. Michal served tea at their home for members of the Society and their friends.

The titles and cross references to abstracts of papers read at the meeting are given below, abstracts whose numbers are followed by *t* having been read by title. Mr. R. C. James, Mr. Knox Millsaps, and Mr. Edmund Pinney were introduced by Professor A. D. Michal, Mr. S. P. Avann by Professor Morgan Ward, Mr. C. B. Barker by Professor C. B. Morrey, Jr., Dr. O. G. Owens by Professor Hans Lewy, and Mr. R. D. Gordon by Professor Max Zorn.

1. G. W. Mackey: *On the lattice of closed linear subspaces of a normed linear space.* (Abstract 47-11-472.)

2. D. H. Lehmer: *Properties of the coefficients of the modular invariant* $J(\tau)$. (Abstract 47-11-457.)

3. R. J. Levit: *Fields in terms of a single operation.* (Abstract 47-11-459.)

4. Morgan Ward: *Conditions for a lattice to be a Boolean algebra.* (Abstract 48-1-23.)

5. R. C. James: *Linearly arc-wise connected topological groups.* (Abstract 48-1-92.)

6. Knox Millsaps: *First order differentials of functions with arguments in topological groups and values in topological abelian groups.* (Abstract 48-1-96.)

7. Edmund Pinney: *Theory of functions on linear topological spaces to Banach spaces.* (Abstract 48-1-50.)

8. S. P. Avann: *The τ-function and its application to semi-modular lattices.* (Abstract 48-1-3.)

9. P. M. Swingle: *Topological foundations of an uncertain mathematics.* (Abstract 48-1-99.)

10. G. F. McEwen: *On the probability that a ratio of random numbers will depart from a harmonic ratio by less than a given amount.* (Abstract 48-1-89.)

11. C. B. Barker: *The Lagrange multiplier rule for two dependent and two independent variables.* (Abstract 48-1-28.)

12. R. D. Gordon: *An application of the Cauchy integral to a problem in probability theory.* (Abstract 48-1-88.)

13. Einar Hille: *On the oscillation of differential transforms.* II. (Abstract 48-1-37.)

14. R. M. Robinson: *Bounded univalent functions.* (Abstract 48-1-54-*t*.)

15. O. G. Owens: *An explicit formula for the solution of the ultra-hyperbolic equation in four independent variables.* (Abstract 48-1-48-*t*.)

16. R. P. Dilworth: *A theorem on lattice homomorphisms.* (Abstract 48-1-10-*t*.)

17. R. P. Dilworth: *Dependence relations in a Birkhoff lattice.* (Abstract 48-1-11-*t*.)

18. Nathaniel Coburn: *Unitary curves in unitary space.* (Abstract 48-1-76-*t*.)

19. R. C. James: *Normable topological abelian groups.* (Abstract 48-1-93-*t*.)

20. František Wolf: *On majorants of analytic functions.* (Abstract 48-1-62-*t*.)

21. S. P. Avann: *Lattices with arbitrary automorphism groups.* (Abstract 48-1-2-*t*.)

22. Gabor Szegö: *On the oscillation of differential transforms.* I. (Abstract 48-1-57-*t*.)

T. M. PUTNAM,
Associate Secretary

BOOK REVIEWS

Diophantische Gleichungen. By T. Skolem. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, Heft 4. Berlin, Springer, 1938. 130 pp. R.M. 15.

Included in the author's preface is this statement, "Hier dagegen ist es versucht worden, eine zusammenfassende Darstellung der Theorie der diophantischen Gleichungen zu geben von den einfachsten bis zu den schwierigsten, die man bis jetzt hat bewaltigen können, aber nur soweit sie allgemeinen Methoden zugänglich sind. Es gibt bekanntlich auf diesem Gebiete sehr viele Untersuchungen über ganz spezielle Gleichungen; auf solche wird nicht eingegangen."

All the equations considered by Skolem, with the exception of several types considered in Chapter II, are of the form

$$f(x_1, x_2, \cdots, x_n) = 0,$$

the left-hand member of the equation being a polynomial in the unknowns $x_1, x_2, \cdots, x_n$ with given integral coefficients, the problem being to find all rational solutions or, in particular, integral solutions. Obviously, not all equations of this type which have been treated in the literature could have been considered in a pamphlet of this size. Hence he has elected, as he states above, to take up mainly what he regards as the most general types of equations concerning which definite results have been found. For example, congruences are special kinds of diophantine equations, but Skolem has wisely omitted any discussion of them, except for certain systems considered on two pages in the first chapter.

Many theorems are stated and proofs of some are given. When the proofs are not given references are usually indicated which enable us to find the demonstrations in the literature.

Diophantine analysis is noteworthy for the great interest which mathematicians have exhibited in certain methods which so far have been applied only to comparatively special equations. A number of these are not mentioned by Skolem, and we think it will be illuminating to refer to them in the course of this review.

In the first chapter of the book, the author considers linear equations of the form

$$\sum_{s=1}^{n} a_{rs}x_s = b_r,$$

$r = 1, 2, \cdots, m$, and gives a number of the results of Heger, Smith

and Frobenius. This work connects up with the theory of the elementary divisors of a matrix.

In Chapter II Skolem treats equations which are linear in each unknown, in particular the bilinear equation

$$\sum_{rs} a_{rs}x_r y_s = 0.$$

In this general theory it is convenient to employ equations in determinant form.

The third chapter treats quadratic equations in two or more unknowns. The homogeneous case is treated first. Smith's criterion for the integral solution of

$$ax^2 + a_1x_1^2 + a_2x_2^2 + 2bx_1x_2 + 2b_1xx_2 + 2b_2xx_1 = 0$$

is given. Proceeding to the equation

$$\sum_{i=1}^{n} a_i x_i^2 = 0, \tag{1}$$

the beautiful theory for the case $n=3$ is treated. Meyer's results on (1) are stated and also this theorem that (1) for $n=5$ is always satisfied in integers not all zero with integers $a_1, a_2, \cdots, a_5$, not zero, and not all of like sign. The first complete proof is by Dickson.[1] Skolem gives Mordell's proof of this after first giving his own proof of Hasse's results concerning the congruence

$$\sum_{i=1}^{n} a_i x_i^2 \equiv 0 \pmod{m},$$

on which the latter shows the solution of (1) depends, if f is indefinite.

Next the nonhomogeneous quadratic equations are treated. He considers the classic theory of representation of integers by ternary forms, but does not mention a number of theories concerning quadratic forms which have so far attracted great attention from many mathematicians, such as the representation of integers as the sum of squares and closely related topics developed by Liouville, Eisenstein, Minkowski, Smith, Bell, Nasimoff, Hardy, Uspensky, Mordell and others, using the methods of elliptic functions, analytic additive number theory, for example. Also, the application of quaternions and related linear algebras to such questions as developed by A. Hurwitz, Dickson and others is not taken up.

[1] Studies in the Theory of Numbers, pp. 68–70.

In the fourth chapter entitled "Multiplikative Gleichungen" the author gives, with the proof, the results of Bell and Ward on the equations,

$$A_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_k^{a_{ik}} = B_i y_1^{b_{i1}} y_2^{b_{i2}} \cdots y_e^{b_{ie}}.$$

He then takes up equations obtained by expanding

$$N(\alpha_1 x_1 + \cdots + \alpha_n x_n)$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are integers in the algebraic field K of degree n, N denotes the norm in K, and setting

$$N = a, \tag{2}$$

where a is a rational integer, and treats by the use of units in K the problem of finding other sets of solutions of (2), having given one set, as well as methods for determining if there exists any set.

He then examines equations of the form

$$f(x_1, x_2, \cdots, x_n) = h y_1^{e_1} y_2^{e_2} \cdots y_m^{e_m},$$

where f is decomposable in some algebraic field. The results of Ward[2] on the equation

$$x^2 - dy^2 = z^m, \qquad d > 0,$$

are given. Ward obtains the primitive solutions of this by use of the field defined by $d^{1/2}$.

Dickson's solution of

$$q = ax^2 + bxy + cy^2 = w_1 \cdots w_n,$$

is mentioned, then a generalization to the equation

$$q = h z_1^{a_1} \cdots z_n^{a_n}$$

is discussed by Skolem. He remarks that the methods he employs extend to the equation,

$$f(x_1, \cdots, x_m) = h z_1^{a_1} \cdots z_n^{a_n}$$

where f is decomposable into m linear factors in some algebraic field, and where the degree of $f > 2$. For the case $h = 1$, $a_1 = \cdots = a_n = 1$, a solution involving the use of ideals was given by Wahlin.

The fifth chapter is concerned with the application of geometry to various types of diophantine equations, when we are seeking rational solutions. It is in this chapter, in the sixth and in the last chapter,

[2] Transactions of this Society, vol. 38 (1935), pp. 447–457.

that it appears the most general results are to be found in the work. The curves

$$f(x, y) = 0, \quad \text{or} \quad f(x, y, z) = 0$$

in homogeneous coordinates are classified for arithmetical purposes according to their genus, the coefficients of the equations being integral. Two curves are called equivalent when they are connected with each other by a birational transformation with rational coefficients. Thus two curves

$$f_1(x, y) = 0, \qquad f_2(\xi, \eta) = 0$$

are equivalent if the coordinates x, y are rationally expressible with rational coefficients in terms of ξ, η, and conversely.

If the curve is of genus zero, it was shown by Poincaré that it is equivalent to a straight line or a conic, from which the theory of the rational points on the original curve is easily derived.

For a curve of genus unity Poincaré also proved that if it has a rational point, it is equivalent to a curve of the third degree, and, in particular, the cubic is equivalent to a curve whose equation can be written in the Weierstrass normal form

$$y^2 = 4x^3 - g_2x - g_3. \tag{3}$$

The latter statement was proved by Mordell in 1912.

There is a well known geometric method for finding rational points on a curve when certain rational points are known initially. If one of the latter points is P (not a point of inflexion), the tangent at P meets the curve in another point P_1 whose coordinates are determined by an equation of the first degree and hence are rational. If Q is another such point, the chord PQ will meet the cubic in a rational point R, in general different from P.

To interpret this process analytically Poincaré and Hurwitz noted that the equation (3) could be given in parametric form by the use of Weierstrass elliptic p functions so that

$$x = p(u), \qquad y = p'(u). \tag{3a}$$

It was proved by Mordell by the use of such ideas that all the rational points on (3) could be found from a finite number of rational points, so that, using (3a), if $u_1, u_2, \cdots, u_n$ satisfy (2) all points are given by $m_1u_1+m_2u_2+\cdots+m_nu_n$, where the m's are integers, that is, the parametric values for all the rational points form an additive Abelian group with a finite basis.

Skolem mentions the work of Fueter and Brunner on the equation

$$x^2 - y^2 = D.$$

They were able to determine the number of solutions of this equation for various values of D, using the field defined by $(-D)^{1/2}$. The author then considers curves of genus $p>1$ and notes that it has been conjectured that such curves contain only a finite number of rational points. A curve of this type is given by the Fermat equation

$$x^n + y^n = z^n.$$

In this connection he mentions only the work of Kummer, Wieferich, Mirimanoff, Furtwangler, and Kapferer, on Fermat's last theorem.

A rational system of points is defined as a finite system of points such that all the symmetric functions of their parametric values are rational. Using this idea Weil showed that Mordell's result concerning the parametric values associated with the rational points on a cubic curve forming an additive Abelian group could be extended to curves of genus $p>1$.

In Chapter VI Skolem goes into detail concerning the theorems associated with the names of Thue and Siegel. These theorems enable us to state that certain equations of the type

$$f(x, y) = c,$$

have only a finite number of solutions. These ideas seem to have had their beginnings in the work of Runge (1887), who proved among other results the following one:

Let $f(x, y)$ be a polynomial with integral coefficients of degree n irreducible in the rational field, the homogeneous part of f of highest degree not being the power of an irreducible polynomial. Then $f(x, y)=c$ has only a finite number of solutions in rational integers.

Thue's theorem of the year 1909 has to do with a polynomial of a type a bit different from that treated in Runge's theorem. It is as follows:

If $f(x, y)$ is a homogeneous irreducible polynomial of degree greater than 2 with integral coefficients and c is a given integer not equal to 0, *the equation $f(x, y)=c$ has only a finite number of integral solutions.*

The author proves this theorem as well as a number of lemmas concerning the approximation to a real algebraic number by the use of a rational number, on which these investigations depend. Using these results it is possible to show that the equation

$$ay^2 + by + c = dx^n,$$
$$a(b^2 - 4ac)d \neq 0, \qquad n \geqq 3,$$

has only a finite number of solutions as well as the equation

$$y^2 = ax^n + bx^{n-1} + \cdots + k$$

where, in the latter, the right-hand member has at least three different zeros.

After considering a number of special equations of the form

$$x^n + dy^n = \pm 1$$

where $n = 3$ or 4, Skolem applies the theory of p-adic numbers to the equation

$$N(\alpha x + \beta y + \gamma z) = h$$

where α, β and γ are integers in an algebraic field K of degree n. He finds equations of this type which have only a finite number of solutions for $n = 5$.

We now signalize a problem which seems fundamental in this subject. If we consider the irreducible equation

$$f(x_1, x_2, \cdots, x_k) = c$$

where f is of degree n with integral coefficients and with c integral and also f homogeneous we know from the theory of units in an algebraic field that for $k = n$ and $c = 1$, there exist equations of this type with an infinity of integral solutions. On the other hand, if $k = 2$, $n > 2$, Thue's theorem states that there cannot be more than a finite number of solutions. The question is, how far must k be increased to obtain equations of this type with an infinity of solutions? If $n = 3$, we have $k = 3$.

The arithmetical theory of Hermitian forms is not considered, likewise Waring's theorem. It is not exactly surprising that the latter topic has been omitted, as it would merit a volume in itself.

Skolem has written a very interesting book. It is surprising how much arithmetical meat he has packed into the space he employs.

H. S. VANDIVER

Sur les Fonctions Orthogonales de Plusieurs Variables Complexes, avec les Applications à la Théorie des Fonctions Analytiques. By Stefan Bergman. New York, Interscience Publishers, 1941. 62 pp. $1.50.

This book was to appear as one of the Mémorial des Sciences Mathématiques series, but circumstances were such that the edition reached only the galley proof stage. The book is a photostatic edi-

tion made from the galley proofs. As a result, the characters are sometimes a little blurred and in some places the lines are uneven.

At the outset the author states that many important results in the theory of one complex variable cannot be extended to functions of several complex variables. To find a technique analogous to that of one variable it is necessary to introduce a new approach. One such approach is the method of orthogonal functions for n variables in a domain B. In this book, however, the author limits himself to the properties of functions of two complex variables. The methods used and the results obtained can be extended to functions of several complex variables. The power of the method of orthogonal functions lies in the following properties of the kernel function $\sum\phi^{(\nu)}(z_1, z_2)\cdot\phi^{(\nu)}(\bar{t}_1, \bar{t}_2) = K_B(z_1, z_2, \bar{t}_1, \bar{t}_2)$, established by the author, of a system of functions closed for the class, $F(B)$, of functions analytic and of summable square in B. First, the function $K_B(z, \bar{t})$, $z \equiv \{z_1, z_2\}$, $\bar{t} \equiv \{\bar{t}_1, \bar{t}_2\}$ is regular in the variables $z_1, z_2, \bar{t}_1, \bar{t}_2$. Second, $K_B(z, \bar{t})$ is independent of the particular choice of the closed system $\{\phi^{(\nu)}\}$. Third, the minimum value of the integral $\int_B |h|^2 dw$, $dw = dx_1 dy_1 dx_2 dy_2$, where h belongs to $F(B)$, and $h(t_1, t_2) = 1$, that is, h is normalized at the point (t_1, t_2), is equal to $[K_B(t, \bar{t})]^{-1} = \lambda_B(t)$, and this minimum is attained for $h = f = K_B(z, \bar{t})/K_B(t, \bar{t})$. It is from these properties that he builds up a theory of orthogonal functions for two complex variables and uses this to deal with analytic functions of two complex variables. This method of the kernel function is very neatly applied with much success to the study of pseudo-conformal transformations. It is extremely interesting that these results can also be applied to the study of linear partial differential equations.

This theory of conformal transformations is essentially based on the Riemannian mapping theorem which states that every simply connected domain possessing at least two boundary points can be mapped conformally into a circle, and the Schwarz lemma. This lemma, as formulated by Pick, states that if a transformation takes the unit circle into its interior, then the hyperbolic length of a line segment included in the unit circle cannot be increased.

Transformations of the domains in four-dimensional space by a pair of analytic functions are called pseudo-conformal transformations. As was first pointed out by Poincaré, two arbitrary simply-connected four-dimensional domains cannot be transformed one into the other by a pseudo-conformal transformation. Also Hartogs has shown that every analytic function regular in a simply connected four-dimensional region assumes on the boundary of this region all the values it assumes in the interior, that is, a function cannot be con-

structed that vanishes at some point in the interior of the domain B and has an absolute value of one on the boundary. An immediate generalization of the Schwarz-Pick lemma therefore cannot be obtained for functions of two complex variables. Therefore, analogous methods to those described above cannot be applied in the case of functions of two complex variables.

To overcome these difficulties the author employs the kernel function $K_B(t, \bar{t})$. In the case of conformal transformations $ds^2 = K_B(z, \bar{z})|dz|^2$ defines a metric which is invariant with respect to conformal transformations. If the domain is simply connected, the metric becomes a hyperbolic metric. Analogously, using the kernel function for functions of two complex variables, the author defines a metric which is invariant with respect to pseudo-conformal transformations. It is clear that if a domain $G \subset B$ then $\lambda_G(t) \leqq \lambda_B(t)$, and since λ is the inverse of the kernel K, $K_G(z, \bar{z}) \geqq K_B(z, \bar{z})$ and hence the metric $K_G(z, \bar{z})|dz|^2 \geqq K_B(z, \bar{z})|dz|^2$. This is a form of the Schwarz-Pick lemma.

The book consists of an introduction and four chapters. In the introduction the author describes his approach to the problem of conformal mapping using orthogonal functions.

Chapter I deals with the characteristics of the space of two complex variables. The author describes domains in the z_1, z_2-space which are of particular importance in the study of analytic functions. In particular, he deals with analytic surfaces, hypersurfaces, and domains with distinguished boundary surfaces. These geometrical studies are not employed in the book, but without an understanding of them one might not fully appreciate the value of the results obtained. These geometrical properties, however, are of fundamental importance in quite another aspect of the theory of functions of two complex variables that the author has developed in other papers. There are several points in Chapter I that would be of particular interest to differential geometers.

Chapter II, which forms the heart of the book, is devoted to the properties of orthogonal functions. The differences between real and analytic orthogonal functions are stressed. The essential and interesting theorem is that the kernel function of complex analytic functions converges at every interior point of the domain. The connection between the theory of orthogonal functions and conformal mapping is described. Special systems of orthogonal functions are also studied.

In Chapter III the author concerns himself with minima problems of the type described above. The problem of interpolation and the application of orthogonal functions to the theory of entire and

meromorphic functions are treated. But here the work is sketchy, and in some places only references to papers are made. Furthermore, the author does not indicate very clearly what role the theory of orthogonal functions plays in the study of entire and meromorphic functions. This, of course, could not very well have been accomplished in the space devoted to this topic.

In Chapter IV some further properties of the invariant metric are given, and the differential geometric properties of the Hermitian metric are described.

A companion book to this one was to be written dealing with the invariant metric. We hope that this interesting book will soon appear in this country.

This book is essentially a collection of results by the author and other people working in this field. It is to be regretted that some of the value of this book, as a book, is diminished by the lack of elaboration. In many places results are merely stated without proof. However, a fairly complete bibliography is inserted at the end, and in all instances of the text, references are made to original papers. The material of this book is new and interesting, and it appears that this field is by no means exhausted. Anyone interested in this field would find it very stimulating to read this book.

ABE GELBART

An Introduction to Differential Geometry with Use of the Tensor Calculus. By Luther Pfahler Eisenhart. Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press. 1940. 10+304 pages. $3.50.

The author's *Differential Geometry of Curves and Surfaces*, which was published in 1909, has seen extensive use. Since that time, the tensor calculus has come to play an important role in Riemannian Geometry and in the Theory of Relativity and the author has considered it desirable to rewrite the differential geometry of curves and surfaces in terms of the tensor calculus. The differential geometry treated in the present book is about equivalent to that in the first half of the 1909 book, with the addition of the concept of parallelism in the sense of Levi-Civita.

There are four chapters. The first is concerned principally with the properties of curves in euclidean 3-space, but includes a definition of the parametric equations of a surface and the envelope of a one-parameter family of surfaces, in order to include consideration of the developables associated with a curve. The material of the chapter is for the most part included in the first two chapters of the 1909 book

and except for changes in notation due to the use of the summation convention, the order and treatment have not been greatly changed. The principal addition is consideration of the theorem that the envelope of a one-parameter family of non-parallel planes is either a cone, a cylinder, or the tangent surface of a curve.

The second chapter is largely devoted to tensor algebra and calculus. Preparatory to the definition of a tensor, transformations of coordinates in euclidean 3-space and the effects on the fundamental quadratic form of such transformations are considered. Then contravariant and covariant vectors, and scalars, are defined and their geometrical significance is discussed. The definition of a tensor and the development of the algebra of tensors follow. The Christoffel symbols and Riemann tensor are introduced; covariant differentiation is defined; and the chapter closes with a classic existence theorem for a system of partial differential equations.

Chapter III deals with the intrinsic properties of a surface, that is, those properties of a surface which depend only on the coefficients of the first fundamental form. In addition to the fundamental concepts of arc length, angle and area, the concepts of Gaussian curvature, geodesic, geodesic curvature and parallelism in the sense of Levi-Civita are defined by intrinsic methods and studied in detail. Additional material includes the notions of isometric nets, isometric surfaces, differential parameters, conformal correspondence and geodesic correspondence.

Chapter IV deals with properties which depend on the imbedding of the surface in space, or upon the coefficients of the second fundamental quadratic form. The principal topics are: the equations of Gauss and Codazzi; principal radii of curvature and lines of curvature; conjugate nets and asymptotic lines; the spherical representation of a surface; tangential coordinates and lines of curvature. The chapter closes with a brief discussion of spherical and pseudo-spherical surfaces, and minimal surfaces.

The book is well organized, well written, and has excellent illustrations. There is an abundance of exercises of all varieties, some of which illustrate the material treated, others of which extend the theory. A student who has absorbed the contents of the book will know the fundamentals of differential geometry and tensor calculus and should find the passage to the study of n-dimensional Riemannian geometry easy.

The reviewer is not certain that the present text offers the most desirable introduction to the differential geometry of surfaces in three-dimensional euclidean space. The extensive use of tensors brings

with it a needed emphasis on the notion of invariance. In the present case it also brings with it an order and an emphasis on symbolic manipulation which tend to obscure the geometry of the situation. For example, the Gaussian curvature is first met as a function depending on the Riemann tensor, which has been previously defined for no very obvious reason. The geometrical significance of the Gaussian curvature becomes clear only much later. As another example, principal directions for a tensor not the fundamental tensor are introduced abruptly and we must wait many pages before it becomes clear as to why anybody thought of them.

The author of a book on differential geometry faces vexing questions as regards analytical rigor. Since functions enter continually in the subject matter, a first question concerns the class (differentiability) of the functions involved. Many authors restrict their considerations to analytic functions, a method which assumes too much but which avoids pitfalls. In the present book we find the following statement on page 3: "Whenever throughout this book we consider any function, it is considered in a domain within which it is continuous in all its variables, together with such of its derivatives as are involved in the discussion." This process of asking the reader to determine the hypotheses of a theorem from the proof can be somewhat misleading. Since the proofs of theorems usually involve other theorems and since these in turn depend on still other theorems, the reader may find it difficult to determine just what his hypotheses are. Since functions are frequently expanded in series, there would appear to be the possibility of confusion between Class C^{∞} and analytic. Since results obtained under the assumption of analyticity are used in the derivation of various theorems, these theorems are valid only under the assumption of analyticity, and yet no statement concerning analyticity appears in the proofs of these theorems. It seems to the reviewer that it would have been a happier situation if in each theorem the necessary class of the functions involved had been stated.

There are a number of other analytical details which appear objectionable to the reviewer, but in view of the fact that Fine's *Calculus* and the author's *Coordinate Geometry* are extensively used as references, it would appear that the attainments of the reader are not assumed to be extensive, and consequently the point of rigor should perhaps not be stressed.

Professor Eisenhart's new book is an interesting addition to the literature on differential geometry. It should prove to be highly useful.

GUSTAV A. HEDLUND

Mathematical Logic. By W. V. Quine. New York, Norton, 1940. 8 +348 pp. $4.00.

Unfortunately, the system of logic presented in this book admits the Burali-Forti paradox. This admission renders the system inconsistent. As a result the book fails in its primary purpose and will need serious revision.

Not all portions of the book are affected by the paradox. In particular, the first three and last of the book's seven chapters can survive unchanged (except for very minor details in Chapter 7). Also, much valuable material can be salvaged from the three affected chapters, notably major portions of §§38–40, 43, 44, 47–52.

Quine's first chapter deals with the propositional calculus, the second with the theory of quantification, and the third with certain aspects of the theory of classes, including the theories of identity and description. The treatment of these subjects is very thorough, with good explanations, so that for these subjects the book can be recommended either as a text or as a reference book.

Quine's seventh chapter contains a new proof of Gödel's theorem on the existence of undecidable propositions. The novel feature of Quine's proof is the following. Let L be a logic with a denumerable number of symbols. We can think of L as a system involving only two symbols, namely "x" and an accent, by replacing the denumerable symbols of L by x, x', x'', x''', $\cdots$. So there is no loss of generality in assuming that L has a finite number of symbols, namely S_1, S_2, $\cdots$, S_n. Let expressions of L be finite sequences of the S's (allowing multiple uses). Let us endow $Mxyz$ with the meaning: if x is a single symbol, then x is the next symbol after y in the list S_1, S_2, $\cdots$, S_n; if x is a complex expression, then x is the result of writing y followed by z; if x is not an expression then $x=y$. Then, as Quine shows in detail, a symbolism based on M, four variables, an accent (for producing more variables), the stroke function, and parentheses (for universal quantification) suffices for the usual syntactical discussions about L. In particular, it suffices to define "provable formula" for L. If we take the S's to be the nine symbols of the M-system, then the M-system suffices for its own syntax, and the Gödel theorem follows readily.

BARKLEY ROSSER

NOTES

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: C. R. Adams, R. P. Agnew, Emil Artin, Reinhold Baer, Harry Bateman, E. T. Bell, P. O. Bell, Garrett Birkhoff, L. M. Blumenthal, R. P. Boas, H. F. Bohnenblust, D. G. Bourgin, A. T. Brauer, Richard Brauer, A. B. Brown, Herbert Busemann, E. W. Chittenden, R. V. Churchill, J. A. Clarkson, George Comenetz, A. H. Copeland, Richard Courant, H. S. M. Coxeter, D. R. Curtiss, M. M. Day, J. J. DeCicco, D. M. Dribin, Samuel Eilenberg, Arnold Emch, Aaron Fialkow, W. W. Flexner, Orrin Frink, H. L. Garabedian, J. W. Givens, Saul Gorn, D. W. Hall, Marshall Hall, O. G. Harrold, Einar Hille, Ralph Hull, Dunham Jackson, Nathan Jacobson, R. D. James, B. W. Jones, Mark Kac, Irving Kaplansky, M. S. Knebelman, R. E. Langer, C. G. Latimer, D. H. Lehmer, Walter Leighton, Norman Levinson, E. R. Lorch, E. J. McShane, Saunders MacLane, W. G. Madow, Szolem Mandelbrojt, Morris Marden, G. M. Merriman, E. W. Miller, W. E. Milne, Deane Montgomery, C. B. Morrey, F. J. Murray, John von Neumann, Ivan Niven, Oystein Ore, Gordon Pall, G. H. Peebles, Sam Perlis, R. S. Phillips, George Pólya, Hillel Poritsky, Anatol Rappoport, W. T. Reid, Arthur Rosenthal, A. E. Ross, A. C. Schaeffer, O. F. G. Schilling, I. J. Schoenberg, I. M. Sheffer, N. E. Steenrod, Otto Szász, Gabor Szegö, J. W. Tukey, J. V. Uspensky, R. J. Walker, H. S. Wall, J. L. Walsh, L. E. Ward, L. R. Wilcox, Aurel Wintner, Leo Zippin, Max Zorn, and Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 49 and 50: Garrett Birkhoff, H. R. Brahana, Richard Brauer, H. E. Buchanan, H. B. Curry, Samuel Eilenberg, C. J. Everett, W. K. Feller, L. M. Graves, M. H. Ingraham, Shizuo Kakutani, M. S. Knebelman, B. O. Koopman, D. H. Lehmer, Hans Lewy, N. H. McCoy, E. J. McShane, H. M. MacNeille, F. J. Murray, B. J. Pettis, George Pólya, G. B. Price, H. A. Rademacher, W. C. Randels, P. V. Reichelderfer, A. C. Schaeffer, Peter Scherk, O. F. G. Schilling, Wladimir Seidel, J. J. Stoker, M. H. Stone, D. J. Struik, J. D. Tamarkin, J. M. Thomas, and Louis Weisner.

The summer session arranged at Brown University for advanced instruction and research in mechanics, and for which sixty students

were accepted, met with such general approval that the program is being continued during the academic year 1941–1942. Courses are being given in (1) Advanced Dynamics, (2) Elasticity, (3) Fluid Dynamics, (4) Theory of Airflight, (5) Graphical and Numerical Methods in Applied Mathematics, (6) Partial Differential Equations; in addition there are two seminars. More than thirty persons are enrolled. On the staff are Professors J. L. Synge, Richard von Mises, J. D. Tamarkin, Willy Feller, Stefan Bergman, and Willy Prager. The instruction is supported by the Engineering, Sciences, and Management Defense Training Program of the United States Office of Education and by the Carnegie Corporation of New York. A few small fellowships are still available for the second semester. Those desiring information should communicate with the Dean of the Graduate School, Brown University, Providence, Rhode Island.

Yale University offers each year a number of Sterling research fellowships, some of them open to mathematicians. For the year 1942–1943 the normal stipend will be $1800. Applicants must have received the degree of Doctor of Philosophy or its equivalent from approved universities in the United States or foreign countries. In general, preference is given to applicants of not more than thirty-five years of age, but older persons are eligible. Applications should be received before March 1; blanks may be obtained from the Graduate School office, Yale University, New Haven, Connecticut.

Professor T. C. Esty of Amherst College has been given the title professor emeritus.

Mr. L. M. Klauber, vice president and general manager of the San Diego Gas and Electric Company, received an honorary doctorate of laws from the University of California at Los Angeles.

Mr. W. G. Banks of Centenary College has been promoted to an assistant professorship.

Associate Professor J. L. Barnes of Tufts College has been promoted to a professorship.

Dr. P. G. Bergmann has been appointed to an assistant professorship at Black Mountain College, Black Mountain, North Carolina.

Assistant Professor J. G. Bowker of Middlebury College has been promoted to an associate professorship.

Assistant Professor Fannie W. Boyce of Wheaton College, Wheaton, Illinois, has been promoted to an associate professorship.

Dr. J. R. Britton of the University of Colorado has been promoted to an assistant professorship.

Assistant Professor E. A. Cameron of the University of North Carolina has been promoted to an associate professorship.

Dr. Harold Chatland of Montana State University has been promoted to an assistant professorship.

Dr. Max Coral of Wayne University has been promoted to an assistant professorship.

Associate Professor H. B. Curry of Pennsylvania State College has been promoted to a professorship.

Miss Rachel Davison of Houghton College, Houghton, New York, has been promoted to an associate professorship.

Dr. Douglas Derry of the University of Saskatchewan has been promoted to an assistant professorship.

Assistant Professor J. L. Doob of the University of Illinois has been promoted to an associate professorship.

Assistant Professor E. D. Eaves of the University of Tennessee has been promoted to an associate professorship.

Mr. W. F. Eberlein is working as propellor research analyst in the Navy Department.

Associate Professor Nat Edmonson of the Agricultural and Mechanical College of Texas has been promoted to a professorship.

Associate Professor P. D. Edwards of Ball State Teachers College has been promoted to a professorship.

Dr. C. H. Frick of Montana State College has been appointed professor of mathematics at Mary Washington College, Fredericksburg, Virginia.

Assistant Professor D. F. Gunder of the Colorado State College of Agriculture and Mechanic Arts has been promoted to an associate professorship.

Dr. Mary E. Haller of the University of Washington has been promoted to an assistant professorship.

Mr. George Hartnell of the Cheltenham Magnetic Observatory has retired.

Assistant Professor M. A. Heaslet of San José State College has been promoted to an associate professorship.

Miss Gertrude Hendrix of Eastern Illinois State Teachers College has been promoted to an assistant professorship.

Dr. P. G. Hoel of the University of California at Los Angeles has been promoted to an assistant professorship.

Mr. C. W. Hook of Georgia School of Technology has been promoted to an assistant professorship.

Dr. S. B. Jackson of the University of Wisconsin has been appointed to an assistant professorship at the University of Maryland.

Associate Professor G. R. Livingston of San Diego State College has been promoted to a professorship.

Dr. E. R. Lorch of Barnard College, Columbia University, has been promoted to an assistant professorship.

Associate Professor W. G. McGavock of Davidson College has been promoted to a professorship.

Associate Professor H. M. MacNeille of Kenyon College has been promoted to a professorship.

Dr. G. T. Miller of Purdue University has been promoted to an assistant professorship.

Mr. G. R. Mirick has been appointed principal of the high school at Lincoln School of Teachers College, Columbia University.

Dr. A. P. Morse of the University of California has been promoted to an assistant professorship.

Professor L. R. Perkins of Middlebury College has resigned because of ill health.

Dr. George Pólya who has been at Brown University as visiting professor has been appointed to a visiting professorship at Smith College.

Dr. Willy Prager who recently held a professorship of aeronautical theory at the University of Istanbul and who was formerly director of the Aerodynamical Institute at Göttingen has been appointed to a professorship of mechanical engineering at Brown University.

Assistant Professor A. W. Recht of the University of Denver has been promoted to an associate professorship.

Professor H. W. Reddick of Cooper Union is on leave of absence to direct the Defense Training Institute, Brooklyn, New York. Assistant Professor F. H. Miller has been named acting head of the department of mathematics at Cooper Union.

Dr. H. J. Riblet of Adelphi College, Garden City, New York, has been appointed to an assistant professorship at Hofstra College, Hempstead, New York.

Dr. L. D. Rodabaugh has been appointed to an assistant professorship at Butler University.

Dr. M. A. Sadowsky of Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Raphaël Salem has been appointed to a lectureship at Massachusetts Institute of Technology.

Assistant Professor Nathan Schwid of the Texas College of Mines has been promoted to an associate professorship.

Dr. M. F. Smiley of Lehigh University has been promoted to an assistant professorship.

Assistant Professor C. E. Smith of San Diego State College has been promoted to an associate professorship.

Associate Professor C. E. Springer of the University of Oklahoma has been promoted to a professorship.

Mr. H. S. Stanley has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor W. R. Talbot of Lincoln University, Jefferson City, Missouri, has been promoted to an associate professorship.

Assistant Professor E. W. Titt of the University of Maryland has been appointed to an associate professorship at the University of Texas.

Dr. C. B. Tompkins of Princeton University has been appointed to a visiting lectureship at the University of Wisconsin.

Dr. E. P. Vance of the University of Nevada has been promoted to an assistant professorship.

Dr. C. E. Van Orstrand of the U. S. Geological Survey Department has retired.

Assistant Professor R. W. Veatch of the University of Tulsa has been promoted to an associate professorship.

Associate Professor J. A. Ward of Delta State Teachers College has been promoted to a professorship.

Dr. Alexander Weinstein has been appointed lecturer in the department of applied mathematics at the University of Toronto for the present academic year.

Brother Bernard Alfred Welch of Manhattan College has been promoted to an assistant professorship.

Assistant Professor E. D. Wells of the University of Pittsburgh, Erie Center, has been promoted to an associate professorship.

Dr. John Williamson of Johns Hopkins University has been appointed to an associate professorship at Queens College.

Professor R. S. Burington of Case School of Applied Science has been given leave of absence to continue work in the Bureau of Ordnance of the Navy Department.

Dr. F. W. Dresch of the University of California has been granted leave of absence for 1941–1942 for military service and is with the Bureau of Ordnance, Navy Department.

Associate Professor H. T. Engstrom of Yale University is on leave this year to serve in the Navy Department.

Professor H. H. Ferns of the University of Saskatchewan has been granted leave of absence to serve in the Canadian Officers Training Corps.

Assistant Professor A. L. Foster of the University of California has been granted leave of absence for the spring of 1942.

Dr. Marian W. Thornton is on leave of absence from the Minnesota State Department of Health.

Dr. I. E. Segal has been appointed research assistant at Princeton University.

The following appointments to instructorships are announced: Acadia University: Dr. D. S. Miller; Ground School, Air Corps Advanced Flying School, Kelly Field, Texas: Dr. E. G. H. Comfort; University of Alabama: Dr. Albert Neuhaus; Brown University: Dr. K. L. Nielsen; University of California: Dr. R. W. Shephard; Case

School of Applied Science: Dr. R. H. Sorgenfrey; College of the City of New York: Dr. Olaf Helmer; University of Connecticut: Dr. C. G. A. Nordling; De Paul University: Mr. A. R. Craw; Frostburg State Teachers College, Frostburg, Maryland: Dr. W. N. Hallett; Harvard University: Dr. R. S. Phillips; Hunter College: Dr. W. W. Gutzman; University of Illinois: Dr. H. M. Schwartz; Indiana University Extension, East Chicago, Indiana: Mr. H. E. Burns; James Millikin University: Dr. W. W. Denton; Johns Hopkins University: Dr. J. D. Bankier, Dr. L. I. Wade; University of Kansas: Dr. R. S. Pate; Louisiana State University: Dr. P. H. Anderson; Macalester College, St. Paul, Minnesota: Dr. František Wolf; Michigan State Normal College, Ypsilanti, Michigan: Dr. Edith R. Schneckenburger; North Carolina State College: Dr. C. F. Strobel; Princeton University; Mr. T. W. Anderson, Dr. Ernst Snapper; Rensselaer Polytechnic Institute: Mr. H. K. Brown; University of Saskatchewan; Dr. Max Wyman; United States Naval Academy: Mr. R. C. Morrow, Mr. W. H. Sears; University of Vermont: Dr. D. T. McClay; Wellesley College: Miss Alberta Schuettler; University of Wisconsin: Dr. Dorothy L. Bernstein.

The death of Dr. B. G. A. Ackermann-Teubner on February 18, 1941, has been reported. He had been a member of the Society since 1907.

Dr. K. D. Swartzel, who was head of the department of mathematics in the University of Pittsburgh from 1922 until his retirement in 1939, died October 30, 1941. For twenty-eight years prior to 1922 he was a member of the department of mathematics in the Ohio State University.

Mr. H. L. Sweet of Phillips Exeter Academy died March 27, 1941.

Professor T. H. Taliaferro of the University of Maryland died September 25, 1941, at the age of seventy years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. A. A. Albert: *Quadratic forms permitting composition.*

It is proved that a quadratic form over a field F permits composition if and only if it is equivalent in F to the norm form xx^J of an alternative algebra over F such that $x+x^J$ and xx^J are in F. These norm forms are shown to be quadratic forms in 1, 2, 4 or 8 indeterminates, except for the diagonal norm forms (in 2^t indeterminates) of the purely inseparable fields of degree 2^t and exponent 2 over F of characteristic 2. The result has never before been obtained for fields of characteristic 2. Indeed it seems only to have been given completely for algebraically closed fields of characteristic not two. (Received October 20, 1941.)

2. S. P. Avann: *Lattices with arbitrary automorphism groups.*

There exists a partially ordered set having an arbitrary group as its group of automorphisms. Hence there exists a distributive lattice with a given automorphism group. A partially ordered set with cyclic group C_n has at least 3 conjugate sets of n elements for $n \leqq 7$ and at least 2 for $n > 7$. From these facts minimal partially ordered sets, lattices, and distributive lattices with cyclic automorphism groups can easily be obtained. (Received October 25, 1941.)

3. S. P. Avann: *The τ-function and its application to semi-modular lattices.*

Let $\tau[L]$, $\sigma[L]$ be the number of join irreducibles and the number of meet irreducibles, respectively, in a finite lattice L. In a non-distributive Birkhoff lower semi-modular (ξ'') lattice $\sigma > \tau \geqq \rho$, where ρ is rank of lattice. In a finite lattice L the following conditions are equivalent: (1) every element has unique irredundant join irreducible representation; (2) the sublattice generated by the elements covered by an element is a Boolean algebra, for each element of the lattice; (3) L is a finite Jordan-Dedekind chain lattice with $\tau = \rho$; (4) L is a ξ'' lattice with every modular sublattice distributive; (5) L is a ξ'' lattice with no modular non-distributive sublattice with coverings; (6) L is a ξ'' lattice with no modular non-distributive sublattice; (7) L is a ξ'' lattice with $\tau = \rho$. Although some of the equivalences are known, a new method of proof in contrast to combinatorial methods is emphasized. Note (6)→(5) and (3)→(7). (Received October 25, 1941.)

4. Reinhold Baer: *Inverses and zero-divisors.*

It may happen that an element in a ring is both a zero-divisor and an inverse, that

it possesses a right-inverse but no left-inverse, and that it is neither a zero-divisor nor an inverse. Thus there arises the problem of finding conditions which assure the absence of these paradoxical phenomena; and it is the object of the present note to show that chain conditions on the ideals serve this purpose. The methods employed in this investigation allow the existence of universal units to be proved at the same time. (Received November 25, 1941.)

5. I. A. Barnett and C. W. Mendel: *A property of the traces of a square matrix.*

Let A be a square matrix having as elements n^2 independent variables and let $t_1, t_2, \cdots, t_n$ be the traces of $A, A^2, \cdots, A^n$, respectively. Then the jacobian of the t_i with respect to each set of n variables chosen from the original n^2 variables is never identically zero provided only that at least one of the variables is a diagonal element of A. (Received November 18, 1941.)

6. Salomon Bochner and R. S. Phillips: *Absolutely convergent Fourier series in noncommutative normed rings.*

This is a first attempt to extend Gelfand's analysis of normed rings, with $|x \cdot y| \leq |x| \cdot |y|$, from the commutative to the noncommutative case. If R_1 is any (noncommutative) normed ring, if $G = \{m\}$ is any commutative index group, and if R is the normed ring of sequences $A = \{a_m\}$, $a_m \in R_1$, $m \in G$, with the norm $|A| = \sum_m |a_m|$ and the product $A \cdot B = \sum_p a_{m-p} b_p$, then an element A of R has a left-inverse (two-sided inverse) if and only if for each character $\chi(m)$ on G the element $\sum_m a_m \chi(m)$ has a left-inverse (two-sided inverse) in R_1. The familiar case of $\{m\}$ being the additive group of all integers can also be treated very easily by Wiener's original method. (Received November 22, 1941.)

7. R. H. Bruck: *Certain numerical invariants of polyadics.*

Let $x(i_1, i_2, \cdots, i_p)$, where each index i_α ranges over the natural numbers $1, 2, \cdots$, be a set X of indeterminates over a field K, and denote by X_r the subset of X obtained by restricting the indices to the range $1, 2, \cdots, r$. For each r let there be given a form $F_r(X)$ (with assigned coefficients in K) multilinear in each of the p indices $i_1, i_2, \cdots, i_p$ and containing only the indeterminates of X_r; for example, if $p=2$, $F_r(X)$ could be taken to be the determinant $|x_{ij}|$, $i, j = 1, 2, \cdots, r$. The sequence of forms $F_1(X), F_2(X), \cdots$ may be called a rank sequence. Let $A(S_1, S_2, \cdots, S_p)$ be a "polyadic" with components in K, where each label S_j ranges over a finite set of fixed labels and has associated with it a prescribed type of linear transformation. If the identification $x(i_1, i_2, \cdots, i_p) = A(S_{i_1}, S_{i_2}, \cdots, S_{i_p})$ is made, then to each rank sequence corresponds a numerical invariant (or rank) of the polyadic, defined as the first r, possibly infinite, for which $F_r(X) = 0$. The procedure is used to derive most of the known ranks of tensors, (ordinary) matrices and n-way matrices. (Received November 24, 1941.)

8. R. H. Bruck: *Isotopy of algebras with a principal unit.*

In the notation of A. A. Albert (abstract 47-9-331; also *Structure of Algebras*, pp. 9–13) the following, where U, V, W are nonsingular linear transformations, are three equivalent definitions of the isotopy of $\mathfrak{A}_0$ and $\mathfrak{A}$: (1) $R_x^0 = UR_yW$, $y = x^V$; (2) $T_x^0 = VT_zW$, $z = x^U$; (3) $x \circ y = (x^U \cdot y^V)^W$. The present study concerns regular algebras; for example, division algebras and linear closures of finite quasi-groups. $\mathfrak{A}$ is

defined to be regular if it contains a left-regular element a and a right-regular element b; here for example, a is left-regular if the equation $ax=0$ implies $x=0$. $\mathfrak{A}$ has an isotope with principal unit if and only if $\mathfrak{A}$ is regular; the transformations which yield such isotopes are of form $U=PR_b^{-1}$, $V=PT_a^{-1}$, $W=P^{-1}$. If $\mathfrak{A}$, $\mathfrak{B}$ are isotopes with principal units and $\mathfrak{A}_1$ is an invariant subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ contains an invariant subalgebra isotopic to $\mathfrak{A}_1$. Every isotope of a simple algebra with principal unit is a regular simple algebra, but not conversely. In the present paper various applications are considered. It is noted that the isotopes of a division algebra are division algebras, and that a division algebra of order two is isotopic to a field. (Received November 24, 1941.)

9. I. S. Cohen: *A generalization of Macauley's theorem on unmixed ideals.*

Let $\mathfrak{o}$ be the ring of nonhomogeneous coordinates of the general point of an r-dimensional algebraic variety V over a ground field of characteristic zero. Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$, let $\mathfrak{J}=\mathfrak{o}_\mathfrak{p}$ be the quotient ring of $\mathfrak{p}$, and let $\eta_1, \cdots, \eta_s$ be s elements in $\mathfrak{J}$ such that the ideal $\mathfrak{A}=\mathfrak{J}\cdot(\eta_1, \cdots, \eta_s)$ is of dimension less than or equal to $r-s$. It is proved that if $\mathfrak{p}$ represents a simple subvariety of V, then $\mathfrak{A}$ is unmixed, of dimension $r-s$. This result, which is of a local character, implies the following theorem in the large: If an ideal $\mathfrak{B}=\mathfrak{o}\cdot(\zeta_1, \cdots, \zeta_s)$ in $\mathfrak{o}$ is of dimension less than or equal to $r-s$, then any imbedded prime ideal of $\mathfrak{B}$ represents a singular subvariety of V. In the case where V is a linear space (that is, $\mathfrak{o}$ a polynomial ring) these theorems reduce to theorems which have been proved by van der Waerden and Macauley, respectively. (Received November 21, 1941.)

10. R. P. Dilworth: *A theorem on lattice homomorphisms.*

The following theorem is proved: *Let S be a partially ordered set containing a complete, atomic, Boolean algebra B. Then there is an order homomorphism of S onto B leaving B invariant, preserving cross-cuts if they exist, and preserving distributive unions.* For the case in which B contains only two elements, this theorem gives most of the known lattice imbedding theorems. For B finite, this theorem is basic in the study of lattice homomorphisms. (Received October 24, 1941.)

11. R. P. Dilworth: *Dependence relations in a Birkhoff lattice.*

Let P be a set of elements. A relation $\perp$ between elements and subsets of P is a *dependence* relation if (1) $p\perp S+p$, (2) $p\perp S$ and $S\perp T$ implies $p\perp T$, (3) $p\perp S+p'$ implies either $p\perp S$ or $p'\perp S+p$. If P is the set of points of a Birkhoff lattice L it is well known that $p\perp S$ if and only if $p\subseteq \Sigma S$ defines a dependence relation with the above properties. Now let P_k be the elements in L of rank k. Define $r(x)=\rho(x)-k+1$. A subset S of P_k is *independent* if $r(\Sigma T)\geqq n(T)$ for every subset T of S. $n(T)$ denotes the number of elements in T. Define $p\perp S$ if and only if an independent subset T of S exists such that $p+T$ is not independent. It is shown that $\perp$ so defined satisfies (1), (2), and (3). Furthermore the independent sets of P_k are characterized in terms of the lattice structure. Also, a number of applications to imbedding problems are given. (Received October 24, 1941.)

12. Marshall Hall and R. P. Dilworth: *The imbedding problem for modular lattices.*

We prove the following: *There are modular lattices of every dimension greater than*

three which cannot be imbedded in a complemented modular lattice. Every modular lattice of dimension three or less is imbeddable. The proofs rest on the fact that a modular lattice having but one independent modular functional can be imbedded in a complemented modular lattice only if it can be imbedded in a projective geometry. (Received November 24, 1941.)

13. B. W. Jones: *The number of classes in related genera of quadratic forms.*

For every prime p, there is associated with any genus g_1, of quadratic forms, another genus g_2. An upper limit for the number of classes in the genus of g_2 is given which depends on the number of classes in g, and the automorphs of the forms in the genus g_1. Under certain conditions the upper limit is reached. (Received October 6, 1941.)

14. Irving Kaplansky and O. F. G. Schilling: *Some remarks on relatively complete fields.*

In a previous paper of one of the authors, the following question arose: Under what circumstances can a field K be relatively complete in two inequivalent valuations? In this note it is shown that a necessary and sufficient condition is that K be separably algebraically closed, a result analogous to F. K. Schmidt's theorem on multiply complete fields. It is also proved that if a field K admits only cyclic extensions and is not relatively complete in any valuation, then no finite extension of K can be relatively complete. (Received November 10, 1941.)

15. C. C. MacDuffee: *On the composition of algebraic forms.*

If F is an algebraic field of degree n and class number h, every integral number of F corresponds to a matrix of the form $x_1S_1+x_2S_2+\cdots+x_nS_n$ under the regular representation, where S_i is an $n\times n$ matrix with rational integral elements. Furthermore, every ideal of the ith class corresponds under the Poincaré correspondence to a matrix of the form $x_1A_1^{(i)}+x_2A_2^{(i)}+\cdots+x_nA_n^{(i)}$, and conversely, every matrix of this form corresponds to an ideal of the ith class. The polynomials $f_1=|x_1S_1+\cdots+x_nS_n|$, $f_i=|x_1A_1^{(i)}+\cdots+x_nA_n^{(i)}|$ are n-ary n-ic forms which admit composition in the sense of Gauss, and under this operation form a group isomorphic with the class group, the norm form f_1 being the identity. (Received November 25, 1941.)

16. Saunders MacLane and O. F. G. Schilling: *Groups of algebras over an algebraic number field.*

Let K/F be the join $K'\cup K''$ of two normal subfields K' and K'' over an algebraic number field F. Suppose that S, S', S'' are the groups of algebras, prime to the discriminant of K/F, which are split by the fields K, K' and K'', respectively. This paper treats the question: When does the hypothesis $K=K'\cup K''$ imply the corresponding relation $S=S'\cup S''$ for the groups of algebras? It turns out that this relation will hold if and only if the Galois groups of K, K', K'' satisfy a certain condition on the distribution of the elements of maximal order. In particular, this condition always holds if K is abelian and in a number of other cases. (Received October 2, 1941.)

17. R. S. Pate: *Functional homomorphisms.* I. Preliminary report.

Two groups G and G' are considered for which a functional homomorphism $f(g)$

of G to G' may be defined such that (1) $f(g)$ is a subset of elements of G', (2) every element of G' occurs in some $f(g)$ and (3) $f(g_1)f(g_2)=f(g_1g_2)$. The set f^{-1} consists of all elements x of G such that $f(x)=f(g)$. The set $f^{-1}(1)$ is an invariant subgroup of G and $f^{-1}(g)$ a coset of $f^{-1}(1)$. The set of all distinct $f(g)$ is a group G'/f which is simply isomorphic to $G/f^{-1}(1)$. If $f(1)$ is an invariant subgroup of G' and $f(g)$ a coset of $f(1)$, $f(g)$ is the customary true homomorphism between G and G'. Certain sets of conditions on $f(g)$ which reduce a functional homomorphism to a true one are considered. A rigid group is one such that every functional homomorphism of any group to it is a true homomorphism. A group A is the product of two groups B and C if A can be constructed from B and C by the well known multiple homomorphism method. Certain theorems concerning functional homomorphisms, rigid groups and products are proved. (Received October 23, 1941.)

18. Samuel Perlis: *Normal bases of cyclic fields of prime-power degree.*

Let Z be a separable, normal extension of finite degree over a field F. If a quantity u and its conjugates form a basis of Z/F, u is said to generate a normal basis of Z/F, and it is known that such a basis always exists. The present paper considers the case in which Z/F is cyclic of prime-power degree $n=p^e$. It is shown that if p is the characteristic of F, u generates a normal basis of Z/F if and only if the trace of u in Z/F is not zero, and all such quantities are easily found in terms of the structure theory of these fields as developed by Albert. This theorem is false if the characteristic of F is not p. Parallel to Albert's theory, the paper next assumes that F contains p distinct roots of unity and again determines necessary and sufficient conditions. The methods used throughout are elementary and make no appeal to the theories of algebras or representations. (Received November 24, 1941.)

19. O. F. G. Schilling: *Normal extensions of relatively complete fields.*

Let F be a relatively complete field with perfect residue class field. The author discusses the Hilbert theory of infinite normal extensions K/F. It is shown in special cases that the algebraic structure of K/F depends essentially on the residue class field and the value group of K. The results of the Hilbert theory are then employed for the solution of existence problems of normal fields K/F with assigned algebraic and arithmetic properties. (Received October 8, 1941.)

20. M. F. Smiley and W. R. Transue: *Metric lattices as singular metric spaces.*

The fact that each pair of elements of a metric lattice (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 41) which are not comparable together with their sum and product form a *pseudo-linear quadruple* (L. M. Blumenthal, *Distance Geometries*, University of Missouri Studies, vol. 13 (1938), p. 48) suggests that the presence of "sufficiently many" pseudo-linear quadruples in a metric space (M, δ) might ensure that lattice operations could be defined in M so that (M, δ) becomes a metric lattice $(M, \delta, <)$. In this note it is shown that, while this statement is not true, the additional assumption of a weak form of either of two five point transitivities of metric betweenness (Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, this Bulletin, abstract 47-5-196) suffices. (Received November 20, 1941.)

21. R. M. Thrall and C. J. Nesbitt: *On the modular representations of the symmetric group.*

Let S_m denote the symmetric group of degree m. If $m<2p$ the order $m!$ of S_m is divisible by only the first power of the prime number p. The general theory of modular representations is fairly well developed in such cases. Nakayama (*On some modular properties of irreducible representations of symmetric groups* I, Japanese Journal of Mathematics, vol. 17 (1940), pp. 89–108; II, ibid., pp. 411–423) has recently determined the irreducible modular characters for $m<2p$. In the present paper the requirement of irreducibility is relaxed. The structure of the regular representation is given in full. It is shown that there are only a finite number of inequivalent indecomposable modular representations and these are all determined. In the final sections specific matrix forms for these representations are computed for the case $m=p$. (Received November 21, 1941.)

22. T. L. Wade and R. H. Bruck: *The characteristic function of a $2p$-tensor.*

Let $A^{(i)}_{(j)} \equiv A^{i_1 \cdots i_p}_{j_1 \cdots j_p}$, $p>1$, be an arbitrary tensor of the type indicated by its indices, and $\delta^{(i)}_{(j)}$ be the numerical tensor defined in abstract 47-5-184. The determinant of $\lambda\delta^{(i)}_{(j)} - A^{(i)}_{(j)}$, a monic polynomial $f(\lambda)$ of degree $N=n^p$ (n the dimension of the coordinate system), is an invariant, which may be termed the characteristic function of $A^{(i)}_{(j)}$. The set, C, consisting of the coefficients of $f(\lambda)$, yields not all, as in the case $p=1$, but only N out of a total of N^2-n^2+1 functionally independent absolute invariants of $A^{(i)}_{(j)}$ (abstract 47-9-335). The members of C are invariants of the connex $A^{(i)}_{(j)} X^{(j)} Y_{(i)}$, in which $X^{(j)}$ undergoes an arbitrary nonsingular linear transformation of N-dimensional space, and $Y_{(i)}$ the contragredient transformation. This property is in fact characteristic of those invariants of $A^{(i)}_{(j)}$ dependent upon the set C. In particular, $X^{(j)}$ may be of the form $x^{j_1}_{[1} \cdots x^{j_p}_{p]}$, where the p vectors $x^i_{[a]}$ undergo different transformations; similarly for $Y_{(i)}$, where $y^{[a}_i$ is contragedient to $x^{i}_{a]}$. For $A^{(i)}_{(j)}$ bisymmetric $f(\lambda)$ factors, the factors being the analogously defined characteristic functions of the simple algebras of which the algebra of bisymmetric tensors is the direct sum. (Received November 24, 1941.)

23. Morgan Ward: *Conditions for a lattice to be a Boolean algebra.*

Let S be a lattice in which if $a \supset c \supset b$, elements w and r exist such that $a \supset w \supset b$, $a \supset r \supset b$, $a = w \cup c$, $b = r \cap c$. Then if every element of S has a unique complement, S is a Boolean algebra. (Received October 22, 1941.)

24. P. M. Whitman: *Note on a certain representation of lattice elements.* Preliminary report.

In a finite lattice L, if a given element A covers exactly r elements, then A can be represented as the join of r or fewer join-irreducible elements. If this L is the lattice of all splittings of some lattice (abstract 46-9-442), then the number cannot be less than r. Under substantial restrictions, the same conclusions can be stated about the lattice of all colonies of cells (of algae, and so on), descended from a certain cell, which precede a chosen colony, where the cells reproduce by subdivision of one cell into two cells. (Received November 24, 1941.)

25. L. R. Wilcox: *Extensions of semi-modular lattices.* III.

The author's result (abstract 47-5-208) is extended to all complemented semi-modular lattices of dimension equal to or greater than 4. The following theorem is proved. Let L be left complemented (abstract 47-9-356) with the further property that $b, c \in L$, $bc \neq 0$ implies $(a+b)c = a+bc$ for $a \leqq c$; suppose also that there exists in L a chain of length 6. Then there exists a complemented modular lattice Λ containing L order-isomorphically and having the properties (a) $a \in L$, $a \neq 0$, $b \geqq a$ implies $b \in L$, and (b) for $a \in \Lambda$, $b \in L$, $a \leqq b$ there exists $c \in L$ such that c is a complement (in Λ) of a in b. Properties (a), (b) characterize Λ uniquely up to isomorphisms. This theorem is a lattice-theoretic generalization of well known imbeddings of affine and hyperbolic spaces into projective spaces. (Received November 24, 1941.)

26. Leonard Carlitz: *q-Bernoulli numbers and polynomials.*

Rational functions of q are defined by means of $q(qb+1)^m = b^m$ $(m>1)$, where after expansion b^m is replaced by b_m; "polynomials" are defined by $b_m(x) = \sum_{\alpha=0}^{m} C_{m,\alpha} q^{\alpha x} [x]^{m-\alpha} b_\alpha$, where $[x] = (q^x-1)/(q-1)$. Many of the properties of the ordinary Bernoulli numbers and polynomials are readily extended to these quantities; in addition there are certain formulas in the generalized case that are not easily specialized to the case $q=1$. Among possible explicit formulas for b_m may be mentioned $b_m = \sum_{s=0}^{m} 1/[s+1] \sum_{\alpha=0}^{s} (-1)^\alpha \binom{s}{\alpha} q^{\alpha(\alpha+1-2s)/2} [\alpha]^m$, which leads at once to a generalized Staudt-Clausen theorem: $b_m = \sum_{s=2}^{m+1} N_s(q)/F_s(q)$ $(m>0)$, where $F_s(q)$ is the cyclotomic polynomial and deg $N_s <$ deg F_s. (Received November 24, 1941.)

27. Joseph Lehner: *The Ramanujan identities and congruences for powers of eleven.* Preliminary report.

The author proves the existence of a "Ramanujan identity" for the modulus 11^α $(\alpha \geqq 1)$. For $\alpha = 1, 2$, this identity implies the Ramanujan conjecture: $p(n) \equiv 0 \pmod{11^\alpha}$ if $24n \equiv 1 \pmod{11^\alpha}$. The methods used are those of Rademacher's paper *The Ramanujan identities under modular substitution* (to be published in the American Journal of Mathematics). A modification of Hecke's T-operator is used. This operator is defined as follows: $U_{11}F(\tau) = \Sigma_\lambda F(\tau+\lambda/11)$, λ mod 11. If $F(\tau)$ is a modular function belonging to $\Gamma_0(121)$, that subgroup of the modular group defined by $c \equiv 0 \pmod{121}$, then $U_{11}F$ belongs to $\Gamma_0(11)$. Then it can be expressed as a polynomial in $A(\tau)$, $B(\tau)$, certain well known functions which constitute a basis for $\Gamma_0(11)$. By taking F to be $\eta(121\tau)/\eta(\tau)$, where $\eta(\tau)$ is the well known elliptic modular function of Dedekind, we obtain the desired Ramanujan identity for the modulus 11. Identities for higher powers of 11 are then obtained by a two-fold induction, one for even α, the other for odd α. The possibility of proving Ramanujan's conjecture for higher values of α $(\alpha > 2)$ is being investigated. (Received November 19, 1941.)

Analysis

28. C. B. Barker: *The Lagrange multiplier rule for two dependent and two independent variables.*

Let $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$ be of class C'''' on a closed simply connected region $\overline{G}$ of class C_α''' and minimize (1) $\iint_G f(x, y, z_1, z_2, p_1, p_2, q_1, q_2)\,dx\,dy$ among all pairs of functions $z_1(x, y)$ and $z_2(x, y)$ which coincide on the boundary G^* with $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$, respectively, and which satisfy (2) $\phi(x, y, z_1, z_2, p_1, p_2, q_1, q_2) = 0$ on G; assume that f

and ϕ are of class C'''' in their arguments everywhere. It is proved in this paper that there exists a unique multiplier $\lambda(x, y)$ of class C' on $\bar{G}$ such that $\bar{z}_1$ and $\bar{z}_2$ satisfy the equations $(\partial/\partial x)(f_{p_i}-\lambda\phi_{p_i})+(\partial/\partial y)(f_{q_i}-\lambda\phi_{q_i})=f_{z_i}-\lambda\phi_{z_i}$, $i=1, 2$, provided that the pair $(\bar{z}_1, \bar{z}_2)$ is "quasi-normal" with respect to the equation (2). The quasi-normality requires (i) that $\phi_{p_1}\phi_{q_2}-\phi_{p_2}\phi_{q_1}\neq 0$ on $\bar{G}$ for the pair $(\bar{z}_1, \bar{z}_2)$, (ii) that to any pair of functions ζ_1, ζ_2 which vanish on G^* and satisfy the equation of variation on G shall correspond a 1-parameter family of solutions $z_i(x, y; \mu)$ of (2) which coincide with $\bar{z}_i(x, y)$ on G^* for each μ near zero, which reduce to $\bar{z}_i$ for $\mu=0$, and which have the ζ_i as their variations for $\mu=0$, and (iii) that a certain other rather complicated differential expression not be an exact differential on any rectangle on G. (Received November 3, 1941.)

29. E. F. Beckenbach: *Painlevé's theorem and the analytic prolongation of a minimal surface.*

The following generalization of Painlevé's theorem is obtained. If $x_j(u, v)$, $x_j'(u, v)$, $j=1, 2, 3$, are defined, respectively, in contiguous domains D, D' having a rectifiable arc C of boundary in common; if $x_j(u, v)$, $x_j'(u, v)$ are harmonic and satisfy $E=G$, $F=0$ in their respective domains of definition; if $X_j(u, v)$, $X_j'(u, v)$ denote the direction-cosines of the normals to the minimal surfaces S, S' thus determined; and if the above functions remain continuous on C and satisfy the relations $x_j(u, v)=x_j'(u, v)$, $X_j(u, v)=X_j'(u, v)$, $j=1, 2, 3$, for all points on C; then S, S' are analytic prolongations of each other across C. (Received November 22, 1941.)

30. E. F. Beckenbach and Maxwell Reade: *Mean-values and harmonic polynomials.*

It is shown that a necessary and sufficient condition that, for a fixed $n\geqq 3$, a function $f(x, y)$, superficially summable in a finite domain D, assume at each point P of D a value equal to its areal (or peripheral) mean on every regular n-gon whose center is at P and whose circumscribed circle lies together with its interior in D, is that $f(x, y)$ be a harmonic polynomial of degree at most $n-1$. The familiar characteristic property of harmonic functions in terms of circular mean-values appears as a limiting case. (Received November 22, 1941.)

31. Stefan Bergman: *On operators in the theory of partial differential equations and their application.*

There exists for every equation $L(U)\equiv\Delta U+C(z, \bar{z})U=0$, $z=x+iy$, $\bar{z}=x-iy$, a function $E(z, \bar{z}, t)$ such that every solution U, of $L(U)=0$ can be represented in the form $U=P(f)=\int_{-1}^{+1}Ef(\zeta)d\tau$, $\zeta=z(1-t^2)/2$, $d\tau=(1-t^2)^{-1/2}dt$; f being a suitable analytic function of one complex variable ζ. This representation is applied to the solution of boundary value problems. Let $\phi(z, \bar{z})=c$ be a curve k in the xy-plane. If $E(z, \bar{z}, t)=E^*(z, \bar{z}+\phi(z, \bar{z}), t)$ then $P(f)$ and $H(f)=\int_{-1}^{+1}E^*f(\zeta)d\tau$ assume on k the same values. Thus: if F is the analytic function which assumes on k the given values, V, and $f=h$, is the solution of $H(f)=F$, $U=P(h)$ will be the solution of $L(U)=0$ assuming on k the values V. Using the representation $U=\int_0^\pi\int_{-1}^{+1}E(r, t, v)f(\zeta)d\tau dv$ valid for the solutions of $T(U)\equiv U(x, y, z)+C(r^2)U=0$, $r^2=x^2+y^2+z^2$, an analogous method for the solution of boundary value problems of $T(U)=0$ is obtained. Using the last representation the following residue theorem is proved: there exist to every function U_1 two functions U_2 and U_3, $T(U_k)=0$, such that $\int_C(U_1dx+U_2dy+U_3dz)=0$ if C lies on a sphere with

the center at origin and can be reduced to a point in the regularity domain of U_k. (Received November 24, 1941.)

32. Salomon Bochner and W. T. Martin: *A class of removable singularities in several complex variables.*

Given a function $f(z_1, \cdots, z_n)$ continuous in a region R and analytic in R except possibly on some exceptional set E in R. Under what conditions on the set E does this imply that f is analytic throughout R? This question is investigated. One sufficient condition is found to be that (measure E_ϵ)$/\epsilon \to 0$ as $\epsilon \to 0$ where E_ϵ is the set of points in R whose distance from E is less than ϵ. (Received November 24, 1941.)

33. J. W. Bradshaw: *On a certain class of continued fractions.*

Continued fractions which are equivalent to the series $S_k = \sum n^{-k}$ for several values of k are generalized by the addition of a variable term to each of the partial denominators. Series equivalent to the resulting continued fractions are set up by means of two systems of difference equations. Application is made to the calculation of the remainder of S_k after summing a few terms. Some of the continued fractions considered are very rapidly convergent. For example, it has been possible, by means of one of them, to recompute Stieltjes' thirty-place value of S_3. (Received November 21, 1941.)

34. J. L. Doob: *Topics in the theory of Markoff chains.*

Let the matrix function $p_{ij}(t)$ $(t>0)$ $i, j = 1, 2, \cdots$ satisfy the following conditions: $p_{ij}(t) \geqq 0$, $\sum_i p_{ij}(t) = 1$, $p_{ik}(s+t) = \sum_j p_{ij}(s) p_{jk}(t)$. Then the $p_{ij}(t)$ can be interpreted as the transition probabilities of a stochastic (Markoff) process. The properties of the $p_{ij}(t)$ (especially as $t \to \infty$ and $t \to 0$) and the relations between the Markoff process and the matrix function are examined in detail. For example, necessary and sufficient conditions that the Fokker-Planck differential equations in the $p_{ij}(t)$ hold are formulated in terms of the properties of the state function x_t (taking on the value j if the jth state is assumed at time t). (Received October 30, 1941.)

35. H. J. Ettlinger: *The theory of the Riesz integral.*

In this paper the author defines the definite integral following Riesz by means of step functions and, making use of a simple set of axioms, obtains a characterization of the Riemann and the Lebesgue integrals. Other more general integrals are discussed. (Received November 24, 1941.)

36. Abe Gelbart: *On functions of two complex variables with bounded real parts.*

Let $f(z_1, z_2)$ be regular in the interior of a finite four-dimensional domain D^4, belonging to the class $\mathfrak{D}^4$ of domains equivalent to the bicylinder, and having a bounded real part in D. The author obtains an upper bound for $f(z_1, z_2)$ in terms of $A = \max_{\{z_1, z_2\} \in D} \mathrm{Re} f(z_1, z_2)$, $f(0, 0)$ and the domain only. The upper bound becomes unbounded only when the point $\{z_1, z_2\}$ approaches a two-dimensional surface on the three-dimensional boundary, so that a consequence of the inequality is that a function regular in the interior of the domain D^4 with a bounded real part is regular almost everywhere on the three-dimensional boundary. From this result, an upper bound for $\left| \partial^{m+n} f(z_1, z_2) / \partial z_1^m \partial z_2^n \right|$ is obtained again in terms of A, $f(0, 0)$, and the domain only, for the particular domain bounded by the analytic hypersurfaces $S_1^3 = E[z_1 = re^{i\lambda_1}$, $0 \leqq \lambda_1 \leqq 2\pi]$ and $S_2^3 = E[z_1 = re^{i\lambda_2} + p(re^{i\lambda_2}) z_2]$. (Received November 24, 1941).

37. Einar Hille: *On the oscillation of differential transforms.* II.

G. Pólya and N. Wiener have shown that $V[(D^2-\lambda)f(x)] \geqq V[f(x)]$ where $f(x+2\pi)=f(x)$, $V[g(x)]$ is the number of sign changes of $g(x)$ in $(0, 2\pi)$, $D=d/dx$, and $\lambda>0$. From this point of departure they proved that if $V[D^{2n}f(x)] \leqq N$ for all n, then $f(x)$ is a trigonometric polynomial of degree not greater than $N/2$, as well as other results relating $V[D^{2n}f(x)]$ with the analytical properties of $f(x)$. In the present paper this theory is extended to a large class of linear second order differential operators ϑ. For such operators $V[(\vartheta-\lambda)f(x)] \geqq V[f(x)]$, $\lambda>0$, if $f(x)$ satisfies the appropriate boundary conditions. Further, $\liminf_{n\to\infty} V[\vartheta^n f(x)] < \infty$ implies that $f(x)$ is a finite sum of characteristic functions of a corresponding boundary problem. Included in the discussion are the differential operators associated with the names of Bessel, Hermite, Jacobi, Laguerre, Legendre, and Mathieu as well as those of the classical boundary value problems of the Sturm-Liouville type with analytical coefficients. For the Jacobi and Legendre operators partly sharper results have been proved by other methods by G. Szegö in the first paper in this series. (Received November 22, 1941.)

38. Mark Kac: *On convergence of certain series of functions.*

Let $\phi(x)$ be such that (1) $\phi(x+2\pi)=\phi(x)$, (2) $|\phi(x_1)-\phi(x_2)| \leqq M|x_1-x_2|^\alpha$ $(0<\alpha\leqq 1)$, (3) $\int_0^{2\pi}\phi(x)dx=0$, and let $\{n_k\}$ be a sequence of integers satisfying the gap condition $n_{k+1}/n_k>q>1$ $(k=1, 2, \cdots)$. The following theorem is then proved: If $\sum_{k=1}^{\infty}a_k^2<\infty$, the series $\sum_{k=1}^{\infty}a_k\phi(n_kx)$ converges almost everywhere. (Received November 21, 1941.)

39. R. B. Kershner: *The continuity of functions of many variables.*

It is known that a function of n real variables may be continuous with respect to each of the variables separately in a given region and still have discontinuities. In fact the set of these discontinuities may have full measure and may include all points of an $(n-2)$-dimensional region. In this paper it is shown that the set of discontinuities cannot have Menger dimension greater than $(n-2)$. The principal tool is Baire's theorem and correspondingly a stronger but more involved statement is proved involving the category of the discontinuities. With the assumption of a very little extra smoothness, much less than a Lipschitz condition with respect to the separate variables, it is shown that the discontinuities are nowhere dense. Examples are given indicating to what extent the results given are complete. (Received November 25, 1941.)

40. H. N. Laden: *An interpolation polynomial involving derivatives of a prescribed function.*

In an earlier paper (Duke Mathematical Journal, vol. 8 (1941)), the author considered an interpolation polynomial $F_n(x)$ of degree less than or equal to $4n-1$ which takes preassigned values $y_1, y_2, \cdots, y_n$ at prescribed abscissas $x_1, x_2, \cdots, x_n$ and such that $F_n^{(\nu)}(x_i)=0$ $(i=1, 2, \cdots, n;\ \nu=1, 2, 3)$, especially where the abscissas are zeros of classical orthogonal polynomials and $y_i=f(x_i)$ $(i=1, 2, \cdots, n)$, $f(x)$ being an arbitrary continuous function. Earlier conjectures, that some convergence theorems for Laguerre abscissas are "best possible" are shown to be only partially correct. Also, $F_n(0)$ is investigated for Laguerre abscissas, $F_n(\pm 1)$ for Jacobi abscissas. It is shown, further, that for Newton (equidistant) abscissas on $(-1, 1)$, there exist continuous functions on $[-1, 1]$ such that the corresponding $F_n(x)$ does not converge uniformly to $f(x)$ on $[-1, 1]$ as $n\to\infty$. In addition, the case where $F_n^{(\nu)}(x_i)=y_i$ $(\nu=1, 2, 3;\ i=1, 2, \cdots, n)$ are preassigned not necessarily zero is considered for Jacobi abscissas

and for functions with prescribed moduli of continuity. The methods used åre due to Féjer, Shohat and Szegö. (Received November 24, 1941.)

41. E. J. McShane: *On Perron integration.*

Perron's definition of integral has the advantage of elegance, and yields very simple proofs of many theorems; but the most general theorem on integration by parts has not been proved for the Perron integral except by the detour of proving the equivalence of the Perron integral and the special Denjoy integral. Here a definition of integral is presented which is shown to be equivalent to that of Perron, but has the advantage of permitting a direct proof of the general theorem on integration by parts. The new definition is closely related in form to that of Perron; the distinction is that Perron's major functions are replaced by pairs (right majors and left majors), and analogously for minor functions. (Received November 24, 1941.)

42. E. J. McShane: *The derivative of the indefinite Lebesgue integral.*

This note contains a proof of the fundamental theorem on the derivative of the indefinite Lebesgue integral which (it is hoped) has the advantages of simplicity and of requiring little preparatory material. (Received November 24, 1941.)

43. P. T. Maker: *The Cauchy theorem for functions on closed sets.*

Let $f(z)$ be defined on E, a bounded, closed set in the complex plane and such that every point z is the limit of two sequences of points lying on curves having noncollinear tangents at z. It is shown that if $f(z)$ has a derivative at each point of E, there is a sequence of coverings $\{R_n\}$ of E, and a continuation, $f^*(z)$, of $f(z)$ to the rest of the plane, for which $\lim_{n\to\infty}\int_{R_n} f^*(z)dz=0$ and $\lim_{n\to\infty} mR_n = mE$. Conditions weaker than the existence of the derivative are found which give the same result. (Received November 25, 1941.)

44. E. J. Mickle: *Associated double integral variation problems.*

By generalizing the concept of adjoint minimal surfaces, Haar (Mathematische Annalen, vol. 100 (1928), pp. 481–502) has shown that with every extremal surface of a double integral variation problem $J[z]$ in which the integrand function is of the form $F(p, q)$, there can be associated an adjoint extremal surface which is itself an extremal surface of an associated adjoint variation problem. The results of Haar can be extended so that with every extremal surface of the problem $J[z]$ there can be associated twenty-four surfaces such that these surfaces are themselves extremal surfaces, respectively, of twenty-four associated double integral variation problems. The transformations defining the associated problems form a group of order twenty-four. Analogous results can be obtained for parametric double integral variation problems. (Received November 18, 1941.)

45. G. C. Munro: *Systems of linear differential equations with constant coefficients.*

This paper presents a new, simple and direct treatment of systems of linear homogeneous differential equations with constant coefficients. (Received October 9, 1941.)

46. S. B. Myers: *An existence theorem for a self-adjoint system of second-order, linear, homogeneous differential equations.*

In 1929 Morse (Mathematische Annalen, vol. 103 (1930), p. 66) generalized the

classical Sturm comparison theorem to n dimensions. One consequence of his results is that if $p_{ij}(x)\eta_i\eta_j \geqq A^2\sum\eta_i^2$ and $r_{ij}(x)\eta_i\eta_j \leqq B^2\sum\eta_i^2$ for all $(\eta)=(\eta_1, \cdots, \eta_n)$, then there exists a solution $(y)\not\equiv(0)$ of the system of differential equations $d/dx(r_{ij}y_j')+p_{ij}y_i=0$ vanishing at $x=a$ and vanishing later before or at $x=a+\pi B/A$. It is assumed that r_{ij} and p_{ij} are of class C' and $r_{ij}\eta_i\eta_j$ is positive definite. In the present paper the same result is proved under weaker hypotheses: namely, with the above inequalities replaced by $\sum p_{ii}\geqq nA^2$, $\sum r_{ii}\leqq nB^2$. This result has applications to a problem in differential geometry in the large (Myers, Duke Mathematical Journal, vol. 8 (1941), pp. 401–404). (Received November 21, 1941.)

47. N. M. Oboukhoff: *The historical development of total differential as the principal part of the increment of a function of several variables.*

There is more than formalism in the differential calculus of Leibniz, that has irrevocably been assimilated by mathematics. This is his basic concept of the differential: what Leibniz calls "difference" is actually "differential" in the sense of the principal part of the increment of a function; likewise in Leibniz' hands analysis became a particular system of calculus; Leibniz as logician and mathematician promoted the same form of structure, that of calculus. Also, he used pragmatico-operational approach where logical foundations were not strong enough. We can discern the general outlines of neoclassical analysis in the characteristics of Leibniz' doctrine. The first half of the eighteenth century showed deterioration of the foundations of calculus in contrast to an irresistible impetus of its growth and development of applications. Later, a balance was restored by Lagrange; the foundations of calculus were formulated by him in a kind of synthesis of the two major doctrines, those of Leibniz and of Newton. In the nineteenth century Cauchy, following in the footsteps of Leibniz and Lagrange proved that the total differential could be determined independently of derivatives as the principal part of the increment of a function, although he did not use this term; Weierstrass introduced it. (Received November 24, 1941.)

48. O. G. Owens: *An explicit formula for the solution of the ultrahyperbolic equation in four independent variables.*

By adapting H. Lewy's generalization of the Riemann integration method for a linear hyperbolic equation in two independent variables, an explicit formula is derived which gives the value of the solution of the ultrahyperbolic equation in four independent variables. The formula requires initial values which may be sufficiently differentiated, that is, regular initial values, and also certain Riemann functions. It is shown that there must necessarily be two integro-differential equations which hold for the solution and its normal derivative on the initial surface. These results are carried further in the case of a hyper-plane, where, by means of an example, the equalities are shown to be not identically satisfied. (Received October 22, 1941.)

49. J. F. Paydon and H. S. Wall: *An extension of the Stieltjes continued fraction theory.*

The authors show that if a_n is in the parabola $|z|-R(z)=h/2$, $0<h\leqq 1$, and $\sum|b_n|$ diverges, $b_1=1$, $a_{n+1}=1/b_nb_{n+1}$, $(n=1, 2, 3, \cdots)$, then the continued fraction $1/1+a_2t/1+a_3t/1+\cdots$ converges uniformly on the interior of the cardiod $|t|^2=[|t|+R(t)]/2h$. If $\sum|b_n|$ converges the sequences of even and odd approximants converge uniformly to separate limits. The convergence theorem of Stieltjes (a_n real and positive) appears as the limiting case $h\to 0$. The theorem solves the prob-

lem of "limitar-periodic" continued fractions in important cases, for example, if c is not real and less than or equal to $-1/4$ it is easy to determine a circle with center c which bounds a convergence region for $1/1+a_2/1+a_3/1+\cdots$. Also included is a theorem recently announced by Leighton and Thron (abstract 47-7-308). (Received November 5, 1941.)

50. Edmund Pinney: *Theory of functions on linear topological spaces to Banach spaces.*

This paper deals for the most part with the theory of Banach valued functions defined on a linear topological space. The Riemann integral and the linear topological M-differential are defined, and various of their properties are investigated. Finally, these results are applied in the proof of existence theorems for the abstract differential equation system $df(x, v, u) = F(x, f, dx)$, (where differentiation is with respect to x only), $f(u, v, u) = v$; and for the partial differential equation $g_1(u, v; h)+g_2(u, v; F(u, v, h)) = 0$. When $F(x, y, z)$ is linear in y, it is shown that under the conditions given, $f(x, v, u)$ is linear in v, and differentiable with respect to all three places. (Received October 24, 1941.)

51. G. Y. Rainich: *Factorization of polynomials, in a ring, with application to partial differential equations.* Preliminary report.

Matrices have been used for the purpose of obtaining a system of first order equations equivalent to a partial differential equation of the second order (in the same sense that the Cauchy-Riemann equations are equivalent to the Laplace equation) by Brill, Dirac and others. The procedure consists in expressing a second order linear differential operator as a product of two first order operators with matrix coefficients. With a view of extending these results the present paper expresses a quadratic polynomial with indeterminates as variables as the product of a linear and *another quadratic* polynomial; the coefficient field must for this purpose be extended into a noncommutative ring. This ring is studied geometrically by considering its elements as operators on a vector space; upon introduction of coordinates these operators are expressed as matrices. As special cases we obtain the above mentioned solution of Dirac and another solution which leads to relations previously obtained by Duffin. The structure of the general solution is studied in terms of these two special solutions. A generalization in which the second factor is a polynomial of degree higher than the second is indicated. (Received November 24, 1941.)

52. Maxwell Reade: *Some remarks on subharmonic functions.* Preliminary report.

Continuous subharmonic functions are characterized by the property that they are dominated by their circular averages, or mean-values (T. Radó, *Subharmonic Functions*, Berlin, 1937, pp. 7–8). The author investigates the essentiality of the use of circular averages; he uses general regular n-gonal averages instead, for $n \geqq 3$. A typical result is the following. *If $f(x, y)$ is continuous in a bounded simply connected domain D, and if $f(x, y) \leqq 1/4h^2 \int_{-h}^{h}\int_{-h}^{h} f(x+\xi, y+\eta)d\xi d\eta$, holds for all h sufficiently small, for each point (x, y) of D, then $f(x, y)$ is subharmonic in D.* The converse does not hold, in general, without further hypotheses placed upon $f(x, y)$; a simple counterexample is the (sub)harmonic polynomial $x^4-6x^2y^2+y^4$. However, *if $\Delta f(x, y)$ throughout D,* where Δ is the Laplacian operator, *then $f(x, y)$ is dominated by its square averages, for*

sufficiently small squares. The squares used here are for illustrative purposes only; general n-gons are used in the paper. (Received November 17, 1941.)

53. G. E. Reves and Otto Szász: *Some theorems on double trigonometric series.*

The paper generalizes to two variables the theorems of Cantor-Lebesgue and of Fatou-Denjoy-Lusin on trigonometric series, and two theorems of Szász on absolute convergence of Fourier series. The mode of proof is an adaptation of the methods used in the one variable case. (Received November 10, 1941.)

54. R. M. Robinson: *Bounded univalent functions.*

Let $f(z)$ be regular and univalent for $|z|<1$, $|f(z)|<1$ there, and $f(0)=0$. Using the method of Löwner, a detailed study is made of the inequalities involving $|f'(0)|$, $|z_0|$, $|f(z_0)|$, and $|f'(z_0)|$. A typical result is that if $|f'(0)|$ and $|z_0|$ are given, the largest possible value of $|f'(z_0)|$ is attained for a mapping of $|z|<1$ on the unit circle with a radial slit provided $|z_0|\leqq 1/2$, but not in all cases. Unbounded univalent functions are considered as limits of bounded functions. Let $F(z)$ be regular and univalent for $|z|<1$, with $F(0)=0$ and $F'(0)=1$. Relations between $|z_0|$, $|F(z_0)|$, and $|F'(z_0)|$ are studied. In particular, sharp bounds for $|F'(z_0)|$ in terms of $|F(z_0)|$ are given. A striking result is that if $|F(z_0)|\leqq 1/4$ then $|F'(z_0)|<2.07$, but that if $|F(z_0)|$ has a given value greater than 1/4, no upper bound for $|F'(z_0)|$ exists. (Received October 21, 1941.)

55. H. M. Schwartz: *On sequences of Stieltjes integrals.*

This paper continues the discussion of convergence criteria for sequences of Riemann-Stieltjes integrals $J_n=\int_a^b f dg_n$ of a former paper (*Sequences of Stieltjes integrals*, this Bulletin, vol. 47 (1941), pp. 947–955); it contains a development of sets *NS* of necessary and sufficient conditions for the convergence of $\{J_n\}$ for different subclasses of the class F of functions f for which J_n exist (g_n being assumed to be of bounded variation), and it concludes with an indication of the possibility of applying the results to certain linear operations. From results obtained in the aforesaid paper one can conclude that for the class D of functions of F which have no singularities of the second kind, a *NS* set is given by the requirement that the sequence $\{g_n\}$ be uniformly of bounded variation and converge on all the common continuity points of g_n and on a and b, but it was left an open question whether this set of conditions would suffice for subclasses of F wider than D. It is now shown that the answer to this question is in the negative; thus, for example, for the set of the functions of F which have at most one singularity of the second kind at some fixed point u of (a, b), it is necessary to add to the above conditions the requirement of equicontinuity at u of the set $\{g_n\}$. (Received November 24, 1941.)

56. Otto Szász: *On a theorem of Hardy and Littlewood.*

The theorem under consideration asserts that if an integrable function $f(\theta)=o(\log 1/\theta)^{-1}$, then $\sum_0^n(v+1)^{-1}s_v^*=o(\log n)$. Here $s_0^*, \cdots, s_n^*$ are the values $|s_0|, |s_1|, \cdots, |s_n|$ rearranged in decreasing order, and s_n is the nth partial sum of the Fourier series for $\theta=0$. The author proves the same result under a more general assumption; applications are also given. (Received November 10, 1941.)

57. G. Szegö: *On the oscillation of differential transforms.* I.

Let $f(x)$ be a real periodic function with period 2π for which all derivatives $f^{(k)}(x)$ exist, and denote by $2N_k$ the number of the mod 2π distinct sign variations of $f^{(k)}(x)$. The following theorems hold: (1) If $N_k=O(1)$, $f(x)$ is a trigonometric polynomial. (2) If $N_k<k/\log k$, $f(x)$ is an integral function. (3) If $\rho>1$ and $2N_k<(k/\rho)^{1/\rho}$, $f(x)$ is an integral function of order $\rho/(\rho-1)$. (1) is due to G. Pólya and N. Wiener (cf. a forthcoming paper in the Transactions), (2) and (3) are refinements of certain results of these authors. (2) and (3) are best possible results since it can be shown that the conclusions of (2) and (3) are not necessarily valid if $N_k=O(1)$ or $N_k=O(k^\alpha)$, $\alpha>1/\rho$, respectively. The method used in the present paper is different from that of G. Pólya-N. Wiener. It leads also to a refinement of (2) in which from a certain limitation of N_k the analytic character of $f(x)$ in a certain strip is concluded. Finally, analogous results are obtained by replacing $f^{(k)}(x)$ by $\vartheta^k f(x)$ where $\vartheta=(1-x^2)D^2-2xD$ is Legendre's operator (cf. II of this series by E. Hille). (Received November 22, 1941.)

58. E. W. Titt: *A method for integrating the linear hyperbolic equation in three independent variables.*

This paper is concerned with Cauchy's problem with data given over a surface duly inclined to the characteristic cone. The same choice of coordinate system relative to the characteristic cone is made that the writer has used previously (Annals of Mathematics, (2), vol. 40 (1939), pp. 862–891). A solution of the homogeneous adjoint equation is obtained by integrating the elementary solution twice in a direction exterior to the cone. This quantity has a finite discontinuity in one partial derivative as in the case of Green's function for an ordinary differential equation. Integrating over a region bounded by one nap of a cone and the initial surface a formula is obtained for the unknown integrated over a two-dimensional region. Varying the vertex of the cone in order to obtain a formula for u itself seems simpler than in the case of Volterra's method. (Received November 24, 1941.)

59. W. R. Transue: *Contributions to the theory of subharmonic functions.*

First, by use of the two-constant theorem, it is shown that the maximum of a subharmonic function on a level line, $H=\lambda$, of an harmonic function, H, is a convex function of λ. A variety of facts concerning subharmonic functions are shown to be obtainable from this result. Second, a limitation on the mean of u, logarithmically subharmonic, on the circumference of circles within the unit circle is shown to induce a limitation on the mean of u^2. Finally, it is shown that, by replacing Green's function in the representation method of F. Riesz by $E_p(P, Q)\equiv\log PQ - \log OQ + R[(P/Q)+(1/2)(P/Q)^2+\cdots+(1/p)(P/Q)^p]$, subharmonic functions can be represented in the neighborhood of a point in cases where they do not possess an harmonic majorant in that neighborhood and hence where representation using Green's function is not possible. (Received October 24, 1941.)

60. W. J. Trjitzinsky: *Analytic theory of parametric linear partial differential equations.*

In two recent addresses G. D. Birkhoff brings out the significance of asymptotic developments, in the field of linear partial differential equations, for the domain of quantum mechanical ideas. He introduces formal series and makes a conjecture that 'actual' solutions, of an appropriate type, exist asymptotic to these series. In view of

Birkhoff's earlier work in the asymptotic theory of ordinary differential equations, as well as in consequence of certain considerations of mathematical physics, the truth of this conjecture would offhand appear as rather likely. This fact explains the purpose of the present work—taking a purely mathematical point of view, the author considers linear partial differential equations containing a parameter λ and first establishes existence of formal solutions containing those of Birkhoff as a special case. He then establishes (for second order equations) some general existence theorems, asserting existence of 'actual' solutions, which are functions asymptotic to the formal series. This theory is naturally divided into two parts—one relating to equations of elliptic type, the other referring to those of hyperbolic type. Equations of parabolic type have not been considered in the present work. (Received November 24, 1941.)

61. S. M. Ulam: *A geometrical approach to the theory of representations of topological groups.* Preliminary report.

The theorem of von Neumann on the representations of compact groups by sequences of finite matrices is proved by consideration of geometrical properties of compact convex bodies in the space of continuous (real-valued) functions on a compact space. (The use of Haar measure is avoided.) A number of related results are derived. (Received November 25, 1941.)

62. František Wolf: *On majorants of analytic functions.*

If $f(z)$ is analytic in $|x|<a, |y|<b$ and $|f(z)| \leqq M(x)$ where $\int_{-a}^{a} \log^+ \log^+ M(x)dx < \infty$, then to an arbitrary $\delta>0$ corresponds a ϕ, dependent only on $M(x)$ and δ, but independent on the particular $f(z)$, such that $|f(z)| \leqq \phi$ for $|x|<a-\delta, |y|<b-\delta$. This is a generalization of a result of N. Levinson (*Gap and Density Theorems*, p. 127, Theorem XLIII). The theorem is proved by a method used by the author to prove a generalization of Phragmén-Lindelöf theorem (Journal of the London Mathematical Society, vol. 14 (1939), p. 208) which becomes a corollary of the above theorem. Another corollary is Nils Sjögren's result (Congrés des Mathématiques Scandinaves à Helsinfors, 1938): If $f(z)$ is analytic in the unit circle and such that $|f(z)| \leqq M(\arg z)$ for $1-\epsilon<r<1$ and $\int_0^{2\pi} \log^+ \log^+ M(\theta) \cdot d\theta < \infty$, then $|f(z)| \leqq \phi$ in $|z| \leqq 1-\delta<1$. ϕ does not depend on the particular $f(z)$, but only on $M(\theta)$ and δ. (Received October 25, 1941.)

Applied Mathematics

63. A. E. Engelbrecht: *Circular plates with large deflections.*

The nonlinear system of equations derived by von Kármán is used to obtain a solution for a family of thin circular plates involving radial symmetry and having a uniform moment applied at the periphery. The edge of the plate to which the external moment is applied suffers no displacement normal to the plane of the plate, but is free to move laterally. The solution is effected by expanding the deflection w and the stress function ϕ in terms of a small parameter $\epsilon=h/a$ where h is the plate thickness and a the radius of the plate. By this expansion the nonlinear system reduces to an iterative process for determining the successive terms. Satisfactory numerical results are obtained for the deflection, bending moments and direct planar stresses for plates whose maximum deflection is twice the order of the plate thickness. (Received October 21, 1941.)

64. R. E. Gaskell: *On longitudinal vibrations of a bar.*

A uniform cylindrical bar, built-in at the end $x=0$, has attached to the free end, $x=1$, a weight, W. Each section of the bar has an initial displacement, $f(x)$, and the system is released from rest at $t=0$. The longitudinal vibrations of the bar are found by use of the Laplace transformation in both integral and series form. The solution obtained can be verified, except on characteristic lines, when $f(x)$ is a continuous function with its first three derivatives piecewise continuous and $f^{iv}(x)$ bounded and integrable. (Received November 21, 1941.)

65. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* II. Preliminary report.

A complete transformation theory of the solution of the partial differential equations of mathematical physics in two dimensions was recently considered by the author (abstract 47-5-254). With the aid of the finite Fourier transform, solutions were obtained to the two-dimensional equation of heat conduction under prescribed initial and boundary conditions, when the boundaries were either rectangles or circles. This present paper treats the solution of the same equations when the boundaries are conveniently represented in polar coordinates. Instead of a finite Fourier transform, a finite Hankel transform is employed. The regularity of the finite Hankel transform serves as a condition to eliminate any superfluous boundary elements which do not enter directly into the problem. Some special examples are considered. (Received November 21, 1941.)

66. F. B. Hildebrand: *Note on the integro-differential equation of a problem in the theory of plane stress.*

It has been shown by E. Reissner (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 300–305) that the stress σ in a stiffener of length L and cross-sectional area A, attached to a semi-infinite elastic sheet in a direction normal to the edge of the sheet, and subjected to a concentrated axial force F, satisfies the Cauchy integro-differential equation $\sigma(x)+A(L)\sigma(L)\{(L-x)^{-1}+K(L, x)\} = \lambda \int_0^L [A(\xi)\sigma(\xi)]'\{(\xi - x)^{-1} + K(\xi, x)\}d\xi$, where $K(\xi, x) = B_1x\xi/(\xi + x)^3 + (B_2x+B_3\xi)/(\xi+x)^2+B_4/(\xi+x)$ and λ and the B_i are certain constants depending upon the dimensions and properties of the stiffener and sheet. In the present note, use is made of a classical theorem of Plessner to prove, under certain physically reasonable restrictions on the function $A(x)\sigma(x)$, that a solution of this equation must satisfy the condition $A(L)\sigma(L)=0$, so that the stress is determined by the abbreviated equation $\sigma(x)=\lambda \int_0^L [A(\xi)\sigma(\xi)]'\{(\xi-x)^{-1}+K(\xi, x)\}d\xi$ and the boundary conditions $A(0)\sigma(0) = F$, $A(L)\sigma(L)=0$. (Received November 21, 1941.)

67. J. J. L. Hinrichsen: *Bounds for the libration points in a restricted problem of n bodies.*

Let $n-1$ bodies of equal mass be fixed in the vertices of a regular $(n-1)$-sided polygon and rotate with uniform velocity about their common center of gravity. Consider the motion of an infinitesimal mass, moving in the same plane, attracted by the $n-1$ bodies according to the Newtonian law of attraction. If $\Omega(x, y)$ represents the potential function of the system, bounds are found for the points whose coordinates (x, y) are solutions of $\partial\Omega/\partial x=\partial\Omega/\partial y=0$. For $n>3$, there will exist $3n-2$ such points, one at the center and $n-1$ equally distributed along each of three concentric circles.

They lie on axes of symmetry of the polygon and cannot lie further from the center than $(2n-1)/(n-1)$ of the distance from one of the $n-1$ bodies of equal mass to the center of mass of the other $n-2$ bodies. Approximate positions of the points are obtained for $n=3, 4, 5$. (Received October 25, 1941.)

68. D. L. Holl: *Consolidation of elastic earth layers.*

Biot (Journal of Applied Physics, vol. 12 (1941), p. 155) has established the equations (1) $\nabla^2\overline{U}+k_1$ grad div $\overline{U}-k_2$ grad $\sigma=0$ and (2) $\nabla^2\epsilon=k_3\,\partial\epsilon/\partial t$. For the consolidation of an elastic earth containing pores filled with moisture. The vector $\overline{U}$ is the displacement vector whose div $U=\epsilon$, and σ is the excess hydrostatic water pressure, and k_1, k_2, and k_3 are physical constants. Solutions of (1) and (2) are obtained for a layer infinite in horizontal extent but of finite depth and supported on a rigid surface which is either porous or is impervious to the passage of the imprisoned fluid. Specific solutions are given when the surface loads which cover a rectangular loading area and for axially symmetrical loads are known. (Received October 21, 1941.)

69. Isaac Opatowski: *On the theory of lethal irradiation of microörganisms.* II.

By means of the Laplace transformation three sets of formulas are obtained which give, each one independently of the other one, the vulnerability constants k_i (abstract 47-9-414) and the number of quanta $n+1$ necessary to kill the organism, when the number of killed organisms N_{n+1} is known as a function of the time t. However, the k_i's are uniquely determined only if their relative order of magnitude is known a priori. Two sets of these formulas give the homogeneous product sums $h_r(k_1, \cdots, k_{n+1})$ (D. E. Littlewood, *Theory of Group Characters*, p. 82) whereas a third set gives $h_r(k_1^{-1}, \cdots, k_{n+1}^{-1})$ in terms of the moments of dN_{n+1}/dt. Therefore, the evaluation of k_i's is reduced to the solution of an algebraic equation of degree $n+1$. If $\{k_i\}$ is an arithmetic progression, $N_{n+1}(t)$ reduces to the incomplete beta-function. Biophysically this case corresponds to a progressive destruction of the "sensitive volume" of the microörganism (B. M. Duggar, *Biological Effects of Radiation*, vol. II, p. 1321) by the quanta. The problem here treated is equivalent mathematically to that of finding a differential system of the type stated in the abstract referred to above, which is satisfied by a known function $N_{n+1}(t)$. (Received October 6, 1941.)

70. Willy Prager: *Fundamental theorems of a new mathematical theory of plasticity.*

The main difficulty all attempts at a mathematical theory of plasticity have, so far, had to cope with arises from the assumption that the material will not behave in a plastic manner unless a certain invariant of the stress tensor has reached a given critical value. Two different sets of equations will, therefore, be valid in the plastified and the not yet plastified regions. The problem becomes all the more involved by the fact that the boundary between these regions is not known beforehand but has to be determined so as to secure continuity of stresses. In order to avoid this difficulty, the author has proposed stress-strain relations giving a gradual transition from the elastic to the plastic state (Proceedings of the Fifth International Congress of Applied Mechanics, Cambridge, Massachusetts, 1938, p. 234). These relations have been applied to various problems of plane strain (Revue de la Faculté des Sciences de l'Université d'Istanbul, (A), vol. 5 (1941), p. 215). The present paper contains two variational

principles which, in this new mathematical theory of plasticity, play the same role as the principle of least work and Castigliano's principle do in elasticity. (Received November 24, 1941.)

71. J. L. Synge: *On the theory of the airfoil of finite span.*

The problem of the potential flow past an airfoil of finite span is formulated, and then linearized by the usual approximation based on the assumptions that the thickness and curvature of the airfoil are small. The problem is reduced to the solution of an integro-differential equation, which agrees with that obtained by Mattioli in 1939 for a flat plate of no thickness. (Received November 21, 1941.)

72. Feodor Theilheimer: *The potential of curvilinear distributions.*

This paper deals with potentials which are induced by charges distributed along curve segments. It is assumed that the curves and the charge distributions are rational functions of a suitable parameter. Then it can be shown that there exists for every curve a finite number of transcendental functions $p_k(x, y, z)$ such that the potential $h(x, y, z)$ can be represented in closed form by means of these functions $p_k(x, y, z)$ together with algebraic and theta functions. (Cf. Bergmann, Mathematische Annalen, vol. 101 (1929), pp. 534–558.) If the curves can be represented as rational functions of second degree the functions $p_k(x, y, z)$ are connected in a simple way with the elliptic functions, and tables can be used for actual computation. These results supply a method for approximate solution of certain boundary value problems. This method is here applied to an aerodynamic problem taken from the theory of the finite air wing. (Received November 22, 1941.)

73. R. H. Tripp: *Bending of a thin plate having the form of a parallelogram.*

The Lagrange plate equation $N\nabla^4 w = p(x, y)$ is solved for a thin plate whose boundary is a parallelogram with opposite angles $\pi/4$ and $3\pi/4$. Two types of loading are considered: (1) a concentrated load at the center of symmetry and (2) a uniformly distributed load. When all the edges are simply supported the problem becomes that of the solution of two linear infinite systems of equations in infinitely many unknowns of the type $x_i + \sum_j a_{ij}x_j = b_i$. The properties of the determinant $|\delta_{ij} + a_{ij}|$ and b_i are such as to insure the existence and convergence of the solution. Derived results for the moments are discussed. (Received October 21, 1941.)

74. Alexander Weinstein: *On the flexural center and the center of twist.*

These concepts have been treated simultaneously by several authors by Saint-Venant's theory of beams. These authors introduced two essentially different definitions of the flexural center and three essentially different definitions of the center of twist. On the other hand, Sothwell has given a descriptive definition of both centers and has outlined a proof of their coincidence provided that one end of the beam is held by rigid constraints, a situation which cannot be met by Saint-Venant's theory. It is shown in the present paper that the coincidence of both centers can be established by using definitions previously employed by the authors who worked with Saint-Venant's methods. (Received November 21, 1941.)

GEOMETRY

75. L. M. Blumenthal: *Metric characterization of n-dimensional elliptic space* $\mathcal{E}_{nr}$. Preliminary report.

The objective is the characterization of the elliptic metric by means of relations between mutual distances of points in certain finite subsets of the space. If δ-supplementation denotes the process by which a semi-metric Σ^* arises from a semi-metric Σ of diameter d upon replacing arbitrary distances pq in Σ by $\delta - pq$, $\delta \geqq d$, and identifying points with zero distance in Σ^*, then $\mathcal{E}_{nr} = \sup_{\pi r} S_{nr}$, where S_{nr} is the metrically convex spherical surface of radius r and dimension n. It is proved that semi-metric Σ is congruently contained in $\mathcal{E}_{nr}$ if and only if $p, q \in \Sigma$ implies $pq \leqq \pi r/2$ and there exists a $\sup_{\pi r}\Sigma$ congruently contained in S_{nr}. A semi-metric m-tuple $p_1, p_2, \cdots, p_m$ with $p_ip_j \leqq \pi r/2$ is congruently contained in $\mathcal{E}_{nr}$ if and only if a symmetric square matrix (ϵ_{ij}), $\epsilon_{ii} = 1$, $\epsilon_{ij} = 1$ $(i, j = 1, 2, \cdots, m)$, exists such that the determinant $|\epsilon_{ij} \cos(p_ip_j/r)|$ has rank not exceeding $n+1$ with all nonvanishing principal minors positive. This puts in algebraic form the determination of congruence indices for the $\mathcal{E}_{nr}$, at least with respect to finite semi-metric sets. (Received October 24, 1941.)

76. Nathaniel Coburn: *Unitary curves in unitary space.*

The question of the existence of an arc length parameter for a unitary curve K_1 imbedded in an n-dimensional unitary space K_n is discussed. First, the familiar formula for arc length element (ds^2) is generalized. Then, it is shown that if and only if K_1 possesses a natural parameter, does an arc length parameter which is an analytic function of the curve parameter exist. In fact, ∞^1 such parameters exist; all have the same absolute value but different moduli. If the curve parameter is real (K_1 reduces to X_1), then again ∞^1 such arc length parameters exist. One and only one of these parameters is real and positive; this parameter is the one commonly associated with X_1 in K_n. The remainder of the paper is concerned with determining those K_n into which can be imbedded various classes of K_1 which possess an arc length parameter (such K_1 are denoted by U_1). The principal result is: If the metric tensor of K_n is not of rank one, then those unitary K_1 which satisfy a system of differential equations of the third or higher order in the parameter are not U_1. (Received October 24, 1941.)

77. N. A. Court: *On the theory of the tetrahedron.*

A "quasi-polar" sphere (Q) may be associated with the general tetrahedron (T) having for center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T). The following two propositions may serve as samples of the many properties of (Q): The "quasi-polar" sphere is coaxial with the circumsphere (O) and the twelve point sphere (L) of (T); The polar reciprocal tetrahedron of (T) with respect to the sphere (Q) is circumscribed about the medial tetrahedron of (T). A second sphere (G) may be related to (T) having for center the centroid G of (T) and for the square of its radius one forty-eighth of the sum of the squares of (T). The sphere (G) is orthogonal to (Q) and is coaxial with (Q), (O), and (L). The four spheres having for centers the vertices of (T) and orthogonal to (Q) cut the spheres having for diameters the respective medians of (T) along four circles lying on the same sphere, namely the sphere (G) of (T). In the special case when the tetrahedron becomes orthocentric, the spheres (Q) and (G) become, respectively, the polar sphere and the first twelve point sphere of the orthocentric tetrahedron. (Received November 21, 1941.)

78. J. W. Peters: *The euclidean geometry of the n-dimensional simplex.*

In this paper theorems associated with the triangle and the tetrahedron are extended to the n-dimensional simplex formed by $n+1$ points in a euclidean space of n dimensions. The centroid and Monge point of the simplex as well as the centroids and Monge points of the faces are defined. The following extension of Mannheim's theorem for a tetrahedron is proved. The $n+1$ planes determined by the $n+1$ altitudes of the simplex and the Monge points of the corresponding faces meet in the Monge point of the simplex. A hypersphere on the centroids of the faces is discussed. It is shown that this hypersphere passes through $3(n+1)$ points associated with the simplex and has a number of properties similar to the nine point circle associated with a triangle and with the twelve point sphere associated with a tetrahedron. (Received November 8, 1941.)

79. J. L. Vanderslice: *Invariant theory of vector pencil fields.*

At each point $(x^1, \cdots, x^n)$ of a space subject to general coordinate transformations is associated a pencil of contravariant vectors, $\xi^i(x^1, \cdots, x^n, u)$. The parameter u of the pencil is normalized to give an invariant metric parameter x^0 associated with each coordinate system x^i and transforming like the gauge variable of generalized projective geometry. The principal result is the discovery of an affine connection with components which are functions of x^α $(\alpha=0, 1, \cdots, n)$ and a method of covariant differentiation of tensor functions of x^α. A study of the equivalence problem then leads to a complete set of tensor invariants for the vector pencil field. (Received November 17, 1941.)

80. André Weil and C. B. Allendoerfer: *A general proof of the Gauss-Bonnet theorem.*

The following generalized Gauss-Bonnet theorem was recently proved independently by Allendoerfer and Fenchel: If a closed Riemann manifold R_n of even dimension can be made a subspace of an euclidean space then $\int K dO = 1/2\omega_n\chi$ where K is the total curvature of the manifold, ω_n is the area of an n-dimensional sphere, and the integration is over the manifold. The present paper removes the restriction that R_n be a subspace of an euclidean space. To do this R_n is subdivided into simplices each of which is small enough to have an isometric euclidean imbedding. The method of tubes is applied to these subdivisions separately, special attention being paid to their boundaries. The terms resulting from the boundaries are found to be intrinsic and drop out when the simplices are reassembled to form R_n. (Received November 26, 1941.)

Logic and Foundations

81. E. C. Berkeley: *Application of symbolic logic to punch card operations.*

This paper discusses the analysis of operations with punched tabulating cards, and also to some extent operations with handwritten cards, as taking place in a large life insurance company, for purposes of valuing policies, computing, recording, and summarizing payments, and so on. The chief instrument of analysis is a system of coding, constructed using symbolic logic and other techniques. The system of coding is exhibited in part, and examples of the coding are given. Some related problems of

practical and theoretical nature are described and discussed. (Received November 21, 1941.)

82. G. D. W. Berry: *On formalizing semantics.*

Two alternative formalizations of semantics may be distinguished. Quine's *Mathematical Logic* (New York, 1940) with modifications, provides the logical and syntactical components of each. In each, designation is primitive, denotation and truth defined. Besides including statements specifying that all designated expressions are elements, that only abstracts designate, and that nothing designates more than one thing, the axioms of both systems include all statements formed from 'If — is designated, then '—' designates it' by replacing the blanks with any constant term. The paradoxes of Grelling, Richard, König, and Epimenides each requires an hypothesis of the form, '— is designated.' In the first system, which has only one designation-relation, each paradox becomes a *reductio ad absurdum* argument for the existence of an undesignated entity. The argument of the Epimenides, in particular, establishes the existence of non-Tarskian statements, or statements equivalent to the denial of their truth. Some axioms of the form '— is designated' are adopted. Others may be added piecemeal, when needed, at the investigator's risk. In the second system, which has infinitely many designation-relations, each statement of this form is an axiom for some appropriate designation-relation. Classes undesignated and statements non-Tarskian relative to one such relation are respectively designated and Tarskian relative to another. (Received November 26, 1941.)

83. Alonzo Church: *On sense and denotation.*

The title is intended to translate Frege's *Über Sinn und Bedeutung*. The *denotation* of a proper name (including descriptions, class abstracts, also sentences as proper names of truth-values) is that of which it is a name. One may *understand* a proper name in the sense of knowing its meaning linguistically, yet not know its denotation. This linguistic meaning is the *sense*. In particular, the sense of a sentence is the proposition. Two proper names coinciding in sense must have the same denotation—although to determine the denotation, given the sense, may require settling a question of extra-linguistic fact. If a constituent part of a proper name is replaced by another having the same sense, the sense of the whole is not altered; if it is replaced by another of the same denotation, the denotation of the whole is not altered, but the sense may be. Thus is solved, in particular, Russell's puzzle about "the author of Waverley." Assuming tacitly that names have only one kind of meaning, the denotation, Russell concluded that a logically sound language cannot employ descriptions: sentences containing descriptions must be reconstrued as mere convenient abbreviations of sentences of a different form. This seems to be tenable but is less elegant that Frege's solution of the same problem. (Received November 21, 1941.)

84. Nelson Goodman: *Sequences.*

The only method hitherto available for defining sequences on the basis of class theory results in identifying a sequence of n components with a certain class n types higher than those components. By this method, the sequence $x_1 \cdots x_n$ becomes: $\hat{z}\{(z\in 1: z \subset x_1 . \vee . x_1 \subset z . x_1 \neq \Lambda) \vee (z\in 2: z \subset x_2 . \vee . x_2 \subset z . x_2 \neq \Lambda) \vee \cdots (z\in n: z \subset x_n . \vee . x_n \subset z . x_n \neq \Lambda)\}$. The method works for a sequence of finite length n in a universe containing at least $n+1$ elements of the type of members of components of the sequence. That the components must be classed can be shown to be no actual restriction. Any component x_k of a sequence Q is defined as: $\hat{z}\{z\in p'(K \cap Q) . \vee . p'(K \cap Q)$

$=\Lambda.z\in s'(K\cap Q)\}$. The above schemata may be supplanted by formal definitions of the class of sequences and of the kth component of any sequence. Every class of classes is correlated to some one sequence by a definable relation we call the "establishment" of the sequence by the class. A class that establishes a sequence having the members of the class, and the null class, as its components is called "self-ordered." (Received November 21, 1941.)

85. S. C. Kleene: *On the interpretation of intuitionistic number theory.*

A closed existential statement $(Ex)A(x)$ can be interpreted from the constructivist standpoint as a partial communication of a more explicit statement which gives a number x such that $A(x)$ holds, together with such further information as is required to complete the meaning of $A(x)$ for that x. Likewise, a closed generality statement $(x)A(x)$ can be interpreted as the assertion of the possibility of describing an effective general method for obtaining, to any given x, such information as will complete the meaning of $A(x)$ for that x. These ideas are made precise in a truth-definition for the class of statements formed from given general recursive predicates of natural numbers by the operations of the predicate calculus. The definition will be used to obtain some metamathematical results for the intuitionistic predicate calculus and number theory. Typical clauses, for closed statements: If $A(n)$ is realized by a, $(Ex)A(x)$ is *realized* by $2^a\cdot 3^a$. $(x)A(x)$ is *realized* by the Gödel number e of a general recursive function $\phi(x)$ such that, for every n, $\phi(n)$ realizes $A(n)$. $A\rightarrow B$ is *realized* by the Gödel number e of a partial recursive function $\phi(x)$ such that, whenever a realizes A, $\phi(a)$ realizes B. Then A is *realizable*, if there is a number a which realizes it. (Received November 21, 1941.)

86. Barkley Rosser: *The Burali-Forti paradox.*

Four basic principles of the theory of ordinal numbers and well-ordered series are: (1) To every well-ordered series there corresponds a unique ordinal number. (2) The series of ordinal numbers is well-ordered. (3) If x is a term of a well-ordered series S, then the series consisting of all terms of S which precede x is also well-ordered, and has a smaller ordinal number than S. (4) Any ordinal number, α, is the ordinal number of the series of all ordinals which precede α. The Burali-Forti paradox is the statement that these four principles are incompatible. To avoid the paradox, two devices have been tried. One is to adopt some form of a theory of types which will invalidate (4). The other device has the effect of invalidating (1) in certain critical cases. In his book, *Mathematical Logic*, Quine proposed a device which is essentially a combination of the two aforementioned devices. Unfortunately Quine's device does not invalidate (4) and fails to invalidate (1) at certain critical points, and so the Burali-Forti paradox appears in Quine's book. This is shown in detail in the present paper. (Received November 24, 1941.)

87. A. R. Schweitzer: *On the genesis of the algebra of logic in the foundations of geometry.*

Following his theory of geometrical relations (American Journal of Mathematics, vol. 31 (1909), p. 400), the author separates descriptive n-space ($n=1, 2, 3, \cdots$) into 2^{n+1} compartments $C(\ )$, $C(\alpha_i)$, $C(\alpha_i\alpha_j)$, $\cdots$, $C(\alpha_1\alpha_2\cdots\alpha_{n+1})$ by means of two n-simplexes T_1 and T_2 such that $T_1=\alpha_1\alpha_2\cdots\alpha_{n+1}$ with sides prolonged is in the interior of, and cut off by, T_2. Then $\sum(I)$ denotes the reflexive formal sum of all C's; $\sum(\alpha_1)$ is

the sum of all C's containing α_1 as argument, and so on; $\sum(\lambda_1)$ is the sum of all C's not containing α_1, and so on. Then $C(\ )=\sum(\lambda_1\lambda_2\cdots\lambda_{n+1})$, $C(\alpha_1)=\sum(\alpha_1\lambda_2\cdots\lambda_{n+1})$ and so on. The C's are replaced by their corresponding expressions in terms of $\sum$ and the subsequent development follows that of an earlier paper reported in this Bulletin (abstract 47-9-430). The preceding provides a representation R_n for the latter exposition; R_2 is equivalent to the so-called Euler diagram which in turn is topologically equivalent to the figure of a triangle in a descriptive plane extended in accordance with the author's system ${}^2K_2$ (ibid., p. 395, Axiom 12). R_2 is topologically equivalent to the figure of a trifoliate curve inscribed in a circle. Another basis for the algebra of logic in the author's theory is found in a simple modification of his axioms for a linear "permutation" (ibid., p. 370). (Received November 24, 1941.)

Statistics and Probability

88. R. D. Gordon: *An application of the Cauchy integral to a problem in probability theory.*

The author was consulted about interpretations of certain data on biological populations. The problem aims at obtaining error corrections on the data. The classical basis of the Bayes-Laplace formula is chosen: $(x)(n/x)=(n)(x/n)$ where the notation has the usual meaning in probability theory. The probability (n/x) is known through its generating function. The distribution (x) of the parameter x, an integer, is taken on reasonable grounds to be $(x)=1/b+1$, for $0\leqq x\leqq b$; $(x)=0$ otherwise. Then $(n)=\sum(x)(n/x)$ is determined. An "observation" results in a value n_0 of the stochastic variable n. The problem: to determine the distribution $(n/n_0)=\sum(n/x)(x/n_0)$. We actually obtain the generating function for the *moments* of (n/n_0). Simple manipulations result in the Cauchy-integral equation $(x)G_{(n/x)}(t;\ x)=1/2\pi i\int_{(\Gamma)}G_{(n)}(t/s)\cdot E(s;x)ds/s$ where the G's represent generating functions for the probability distributions indicated in their subscripts, and $E(s;\ x)$ represents an *unknown* generating function for probabilities (x/n) *with respect to* n. The problem is solved by applying standard generating-function operators to this equation and integrating over the circle $|s|=t$. (Received November 14, 1941.)

89. G. F. McEwen: *On the probability that a ratio of random numbers will depart from a harmonic ratio by less than a given amount.*

The ratio of two numbers is called harmonic if it equals the ratio of two small integers, one figure numbers, for example. In practical applications both numerator and denominator are subject to error and their ratio accordingly departs from perfect harmony. Accordingly, it is necessary to estimate the probability P of getting by chance a ratio departing less than a given amount from perfect harmony. This problem of determining the proportion, among all possible combinations of numbers, that have a ratio departing from perfect harmony less than this amount, is equivalent to that of finding the probability P that the ratio of any two numbers drawn at random will depart by an amount X or less from the harmonic ratio N/M. The solution is $P=2(M^2/N)X/[1-(M/N)^2X^2]$, where $N>M$. (Received October 27, 1941.)

Topology

90. W. W. Flexner: *Noncommutative chains.* II. Preliminary report.

Extending the author's recent work (Duke Mathematical Journal, vol. 8 (1941), pp. 497–505) to a finite convex 3-complex, 2-dimensional noncommutative chains and

a "homology group" Π_2 are defined. The 2-chains C are products of basic 2-chains $\Omega = aEa^{-1}$ where E is a 2-cell or its inverse and a is a noncommutative one-chain such that $F\Omega = a(FE)a^{-1}$, where FE is the boundary of E, is a continuous closed "path" beginning and ending at a fixed point ω. The group of 2-chains is subject to the relations $CC'C^{-1} = (FC)C'(FC)^{-1}$. The subgroups $\mathfrak{Z}_2$, of cycles, and $\mathfrak{J}_2$, of bounding cycles, are then defined much as in the commutative case. If $\Pi_2 = \mathfrak{Z}_2/\mathfrak{J}_2$, it is commutative and subdivision invariant. (Received November 24, 1941.)

91. D. W. Hall: *On a theorem of E. E. Betz.*

In a recent paper (American Journal of Mathematics, vol. 63 (1941), pp. 127–135) E. E. Betz has studied compact locally connected continua M which satisfy the following conditions: (a) no two points of M separate M, (b) every simple closed curve J of M separates M into at least two and at most a finite number of complementary domains. He has obtained a rather general accessibility theorem for these continua. Using the result of Betz the present paper shows that if M be any compact locally connected continuum satisfying (a) and (b) and if G is any open subset of M, then there is an open subset R of M contained in G which is homeomorphic with an open 2-cell. (Received November 25, 1941.)

92. R. C. James: *Linearly arc-wise connected topological groups.*

The definitions of arc-wise connected, locally arc-wise connected, and simply connected topological spaces are given in terms of a continuous function with real numbers as arguments and values in the topological space. For topological groups, one can add the restriction that the function be linear (additive and continuous) and obtain definitions of linearly arc-wise connected and locally linearly arc-wise connected topological groups. In this paper certain of these properties are used as conditions for a topological abelian group to be a linear topological space. It is shown that a topological abelian group which is simply connected and locally linearly arc-wise connected either has no nonzero elements of finite order or a dense set of elements of the nth order (for each $n > 1$). Several necessary and sufficient conditions for a topological abelian group T to be a linear topological space are found, two of which are: (1) T be connected, locally linearly arc-wise connected, and have no nonzero points of finite order; (2) T be simply connected, locally linearly arc-wise connected, and have a neighborhood of the identity which contains no elements of the second order. (Received October 24, 1941.)

93. R. C. James: *Normable topological abelian groups.*

Several equivalent definitions have been given of bounded sets and convex neighborhoods of a linear topological space. In this paper these concepts are generalized to topological groups, and are used with linear arc-wise connectedness (connectedness by paths defined by a linear function of a real variable) to find conditions that a topological abelian group be normable and that it be a linear topological space. It is shown that a convex, linearly arc-wise connected topological abelian group which has no nonzero elements of finite order is a convex linear topological space, and that a normable, linearly arc-wise connected topological abelian group is a normable linear topological space. A theorem of Komogoroff's on the normability of linear topological spaces is generalized to linearly arc-wise connected topological abelian groups. (Received October 24, 1941.)

94. Saunders MacLane and Samuel Eilenberg: *A theorem on group extensions.*

An abelian group extension of a group G by an abelian group H is customarily represented either by means of a factor set of H in G, or by means of a multiplication table for the generators of H. The latter depends essentially on the representation of H as a factor group $H=F-B$ of a free group F. The exact equivlaence of these two descriptions of a group extension turns out to be important in the study of homology groups (abstract 47-11-487). It may be stated as follows: If F is a free group, and $H=F-B$, then the group Fact $\{G, H\}$ of (abelian) factor sets of H in G modulo the transformation quantities is isomorphic to the group Hom $\{B, G\}$ of all homomorphisms of B into G, modulo the subgroup of the homomorphisms of B into G which can be extended to F. If G is topological, so is this isomorphism. (Received November 24, 1941.)

95. Harlan C. Miller: *On irreducible continua.*

This paper is concerned with properties of compact continua which are irreducible between some two points, and especially of those which are unicoherent. Some of the results are: If M is a compact atriodic unicoherent continuum, and H and K are proper subcontinua such that $M=H+K$, then H and K are unicoherent. If M is a compact hereditarily decomposable continuum irreducible between some two points, then M contains two mutually exclusive continua, H and K, such that (1) in order that M be irreducible between two points it is necessary and sufficient that one of the points belong to H and the other to K, and (2) $M-(H+K)$ is strongly connected. If M is a compact hereditarily decomposable continuum irreducible from point A to point B, there exists an upper semi-continuous collection, G, of mutually exclusive continua filling up M such that if H and K are the elements of G containing A and B, respectively, then with respect to its elements, G is an arc from H to K; in order that M be unicoherent it is necessary and sufficient that G be a continuous collection. (Received November 25, 1941.)

96. Knox Millsaps: *First order differentials of functions with arguments in topological groups and values in topological abelian groups.*

A first order differential is defined for a function $F(x)$ with arguments in a topological group G and values in a topological abelian group A and increments in the central subgroup of G with a relativized topology and a generation postulate. The fundamental theorems on uniqueness for increments in the central subgroup, continuity, linear combinations, and the iterative function $K(F(x))$, where $K(y)$, has arguments and values in t.a.g.'s and is M_1-differentiable (see Michal, Revista de Ciencias), are then proved. (Received October 24, 1941.)

97. M. E. Shanks: *Irreducible transformations of the discontinuum into continua.* Preliminary report.

A continuous transformation T of the Cantor discontinuum C into a continuum is strongly irreducible on a closed subset homeomorphic to C. Irreducible transformations of C into locally connected continua are studied in particular with special reference to the one-dimensional case. Certain results are obtained which indicate a simple structure for T in special one-dimensional cases. (Received October 24, 1941.)

98. R. H. Sorgenfrey: *Some theorems on co-terminal arcs.*

The following results have been established: If H denotes the sum of two arcs with common end points and K denotes their common part, then every component of $H-K$ is a subset of some simple closed curve lying in $\overline{H-K}$. If three arcs have a common end point and no one of them is a subset of any other, then one of them fails to be a subset of the sum of the other two. If three arcs have two end points in common then there are two of them neither of which is a subset of the sum of the remaining two. (Received November 25, 1941.)

99. P. M. Swingle: *Topological foundations of an uncertain mathematics.*

In this paper a probable origin is given to point sets, as in quantum physics, in a universe of incessantly moving electrons. Philosophical consideration is given to the origin of ideas in such a universe and the conclusion reached that there is little reason for the certainty that exists in traditional mathematics. The nature of a probability topology is considered, a degree of certainty being reached in an abstract topology having an interpretation in what is called a diophantine space, whose points are "cube"-units. (Received October 27, 1941.)

100. P. A. White: *R-regular convergence spaces.*

In this paper a hyperspace K_r of a compact metric space M is considered, which consists of all closed locally γ^s-connected $(s \leqq r)$ subsets of M. An infinite sequence $\{p_i\} \subset K$ will be said to converge to $p \subset K$ provided the corresponding subsets $\{P_i\}$ of M converge s-regularly to P, $(s \leqq r)$. This definition of convergence is used to define limit point, closed set, and open set in the obvious manner. It is shown that the original definition of limit point is equivalent to an open set definition, and that K will be a regular Hausdorff space satisfying the first axiom of countability with these open sets as neighborhoods. It is also shown that unless M is finite, K can be written as the sum of a countable number of sets which are both open and closed, and that not more than one of these sets can be compact. (Received November 24, 1941.)

101. G. W. Whitehead: *Homotopy groups of spheres and their rotation groups.*

In this paper relations between the homotopy groups of the n-sphere S^n and those of its rotation group R_n are studied. The relationship between these groups is established as follows: to each map $f(S^m \times S^n) \subset S^n$ there is associated a mapping $f^*(S^{m+n+1}) \subset S^{n+1}$. This operation is a generalization of one introduced by H. Hopf (Fundamenta Mathematicae, vol. 25 (1935), pp. 427–440), and induces a homomorphism of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$. For $m=1, 2$ this homomorphism is an isomorphism. This isomorphism is used to construct a counterexample to a theorem stated without proof by H. Freudenthal (Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 42 (1939), pp. 139–140). In particular, Freudenthal's construction of maps of S^{2n-1} on S^n with Hopf invariant 1 fails if $n \equiv 2 \pmod 4$. (Received November 24, 1941.)

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